ACADEMIC PRESS
Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Functional Analysis 198 (2003) 43–83
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Function spaces on fractals

Robert S. Strichartz[1]

Mathematics Department, Malott Hall, Cornell University, Ithaca, NY 14853, USA

Received 29 October 2001; revised 16 April 2002; accepted 28 April 2002

Abstract

We construct function spaces, analogs of Hölder–Zygmund, Besov and Sobolev spaces, on a class of post-critically finite self-similar fractals in general, and the Sierpinski gasket in particular, based on the Laplacian and effective resistance metric of Kigami. This theory is unrelated to the usual embeddings of these fractals in Euclidean space, and so our spaces are distinct from the function spaces of Jonsson and Wallin, although there are some coincidences for small orders of smoothness. We show that the Laplacian acts as one would expect an elliptic pseudodifferential operator of order $d+1$ on a space of dimension d to act, where d is determined by the growth rate of the measure of metric balls. We establish some Sobolev embedding theorems and some results on complex interpolation on these spaces.

1. Introduction

What are the analogs of the classical function spaces of Hölder–Zygmund, Besov and Sobolev types, when the underlying space is fractal? Here we will be interested in post-critically finite (pcf) self-similar fractals, and the Laplacians and associated effective resistance metrics as described in Kigami's book [Ki]. (See also [Ba] for the probabilistic approach, and [S3] for an informal introduction to the theory.) Although these fractals are conveniently realized as subsets of Euclidean space, the intrinsic analysis on these fractals depends only on their toplogy, and is unrelated to the geometry of the embedding. In particular, the effective resistance metric does not allow an isometric, or even quasi-isometric, embedding in a Euclidean space. Thus the function spaces we construct are quite distinct from the spaces of Jonsson and

E-mail address: str@math.cornell.edu.
[1] Research supported by the National Science Foundation, grant DMS-9970337.

0022-1236/03/$ - see front matter
doi:10.1016/S0022-1236(02)00035-6

Wallin [JW], which are based on Whitney extension ideas, although some identifications are possible for small orders of smoothness.

Our ultimate goal is to generalize the results of Chapter 5 in Stein's book [St] to this context. In the present paper we are able to make a beginning in this program. We give what we believe to be the correct definitions of these function spaces, and we prove most of the basic properties, including Sobolev embedding theorems and results on complex interpolation of these spaces. Some of these results are incomplete for rather technical reasons, and we state some conjectures to fill these gaps. It is also clear that some new ideas will be needed, such as the development of the appropriate Littlewood–Paley theory, in order to complete this program.

In order to describe our results, we outline some of the results of Kigami [Ki]. The reader should consult [Ki] for complete details. Also see Section 7 for a discussion of the general case. The simplest nontrivial example of a pcf self-similar fractal is the Sierpinski gasket SG, generated by the iterated function system (ifs) consisting of the three contractions $F_i x = \frac{1}{2}(x + q_i)$ where (q_1, q_2, q_3) are the vertices of a triangle in the plane. We write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ for any word $w = (w_1, \ldots, w_m)$ of length m, and call $F_w(SG)$ a cell of level m. We have $SG = \bigcup_{|w|=m} F_w(SG)$, the decomposition of SG into cells of level m, and these cells intersect only at points. We call $V_0 = \{q_1, q_2, q_3\}$ the boundary of SG, and let $V_m = \bigcup_{|w|=m} F_w V_0$, the vertices of level m. We form a graph Γ_m with vertices V_m and edge relation $x \sim_m y$ if and only if x and y belong to the same cell. We define an energy

$$\mathscr{E}(u,u) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m y} (u(x) - u(y))^2. \tag{1.1}$$

The limit always exists in $[0, \infty]$, and we define dom $\mathscr{E}$ to be the functions with finite energy. The limit is zero if and only if u is constant, and dom $\mathscr{E}$ modulo constants becomes a Hilbert space contained in the continuous functions modulo constants. We may then define the effective resistance metric

$$d(x,y) = (\min\{\mathscr{E}(u,u) : u(x) = 0, u(y) = 1\})^{-1}. \tag{1.2}$$

The important fact is that $d(x,y)$ is on the order of $(3/5)^m$ for $x \sim_m y$, and the cells of order m have diameter on the order of $(3/5)^m$. Thus $d(x,y)$ is equivalent to $|x-y|^\beta$ for $\beta = \log(\frac{5}{3})/\log 2$.

Let μ denote the standard self-similar measure on SG which assigns measure $(1/3)^m$ to each cell of order m. Then we have the estimate

$$\mu(B_r(x)) \sim r^d \tag{1.3}$$

for $d = \log 3/\log(5/3) \approx 2.1506601\ldots$, where $B_r(x)$ denotes the ball of radius r about x in the effective resistance metric. Thus SG is a space of homogeneous type of dimension d for this choice of metric and measure. There is also a Laplacian Δ associated to the energy and metric. We will see that Δ acts like an operator of order $d+1$. We note that in a lot of the literature in this area it seems to be tacitly assumed that Δ is of order two, in analogy with the usual Laplacians on manifolds. This in turn leads to a different dimension, called the *spectral dimension*. In our view this is a misnomer.

Since we have a metric on SG, we may define Hölder spaces by the condition

$$|u(x) - u(y)| \leqslant Md(x,y)^{\alpha}. \tag{1.4}$$

We will denote this space $\Lambda_{\alpha}^{(1)}(SG)$. In the context of the real line (or more generally for manifolds), it only makes sense to take $\alpha \leqslant 1$. However, it was the truly brilliant observation of Zygmund that for $\alpha = 1$ you obtain the wrong space. What this means is that the Lipschitz space ($\alpha = 1$) does not belong to the same family of spaces as the other Hölder spaces, and the correct space involves replacing the first difference with a second difference, which in turn allows you to take any $\alpha < 2$. For SG we may allow $\alpha \leqslant \alpha_1 \leqslant \log 2/\log(5/3)$ and still obtain a nontrivial space, but again the condition $\alpha < 1$ is required in order to obtain the correct space. To get above this value we use the analog of the second difference, which in this context is the discrete Laplacian. This allows us to go up to $\alpha < \alpha_2 = d + 1$. Using the difference of discrete Laplacians gets us up to $\alpha < \alpha_2 + \alpha_1$, and then we are able to define an entire Hölder–Zygmund scale $\Lambda_{\alpha}(SG)$ for $\alpha > 0$ on which Δ acts by reducing α to $\alpha - \alpha_2$. This construction is carried out in Section 2 of this paper. We conjecture that the Hölder–Zygmund scale is invariant under complex interpolation, and we prove some partial results in this direction.

In Section 3 we discuss Sobelev spaces $L^p(SG)$, for $1 < p < \infty$ and $s \geqslant 0$. When $s = 0$ these are the usual L^p spaces, and when $s = \alpha_2 = d + 1$ these will be the L^p domain of Δ. For other values of s we use fractional powers of $-\Delta$ or $(I - \Delta)$. Certain technical problems arise because we are dealing with an underlying space with boundary. However, since the boundary is finite we are able to get around these difficulties by adding on a finite-dimensional space of multiharmonic functions. The Sobolev spaces are preserved under complex interpolation. This is a routine conclusion from the L^p boundedness of imaginary powers of the Laplacian, a consequence of appropriate heat kernel estimates [BP,HK] and some general spectral multiplier theorems [He,DOS]. We conjecture that the Sobolev spaces embed $L_s^p(SG) \subseteq L^q$ for $s < d/p$, $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$ and $L_s^p(SG) \subseteq \Lambda_{s-d/p}(SG)$ for $s > d/p$, as would be expected in a d-dimensional space. We are able to prove the first statement and most cases of the second. For the second we use special arguments for $s = \alpha_2/2$, $p = 2$ (energy) and for $s = k\alpha_2$ (properties of the Green's function), and then use interpolation. It is not clear whether or not there is any relationship between these Sobolev spaces and the spaces of functions of finite p-energy introduced in [HPS] in the case $p \neq 2$.

In Section 4 we discuss the Besov spaces $\Lambda_{\alpha}^{p,q}(SG)$, which contain in the special case $p = q = \infty$ the Hölder–Zygmund spaces. Here we are compelled to limit our attention to the region $\alpha > d/p$ where the Besov spaces embed in the continuous functions, since we use discrete approximations that would otherwise be meaningless. An important question we are not able to answer is whether or not it is possible to identify $\Lambda_{\alpha}^{2,2}(SG)$ with $L_{\alpha}^2(SG)$.

A more systematic approach to defining Hölder–Zygmund and Besov spaces should involve solving the heat equation with the given function as initial data. In Section 5 we outline what might be expected in this approach, and give a few proofs. In Section 6 we discuss extension and restriction to cells for functions in the function spaces.

In Section 7 we show how the results in the earlier sections may be extended to other pcf self-similar fractals and energy forms. Many of the proofs are easily modified, but in places it seems necessary to make some additional hypotheses, some rather natural and others quite restrictive.

Although the results presented here are often incomplete, we feel that it is important to present them at this time in order to create a coherent context for development of this area, and as a stimulus to further research.

2. Hölder–Zygmund spaces on SG

Our goal is to define the spaces $\Lambda_\alpha(SG)$ analogous to the Hölder–Zygmund spaces on the line (or $\mathbb{R}^n$). First we define the preliminary spaces $\Lambda_\alpha^{(j)}(SG)$ for $j = 1, 2, 3$ in terms of differences of "order" $j = 1, 2, 3$. For certain values of α we will be able to identify $\Lambda_\alpha^{(j)}$ with Λ_α. By identity of Banach spaces we mean of course that the norms are equivalent. All these spaces will be composed of bounded continuous functions, and we will add in the L^∞ norm to all the semi-norms we define to obtain norms in the end. For $\Lambda_\alpha^{(1)}$ condition (2.1) already implies that u is bounded, but this is not true of (2.2) or (2.4). In [DSV] there are examples of unbounded functions that are harmonic away from the boundary, so (2.2) and (2.4) hold with $M = 0$.

Definition 2.1. (a) For $0 < \alpha \leqslant \alpha_1 = \frac{\log 2}{\log 5/3}$ define $\Lambda_\alpha^{(1)}(SG)$ to be the Banach space of all bounded continuous functions on SG satisfying

$$|f(x) - f(y)| \leqslant M\left(\tfrac{3}{5}\right)^{\alpha m} \tag{2.1}$$

for all m and $x, y \in V_m$ with $x \sim_m y$. The norm is the sum of $||f||_\infty$ and the smallest constant M in (2.1).

(b) For $0 < \alpha \leqslant \alpha_2 = \frac{\log 5}{\log 5/3}$ define $\Lambda_\alpha^{(2)}(SG)$ as before with (2.1) replaced by

$$|\Delta_m f(x)| \leqslant M\left(\tfrac{3}{5}\right)^{\alpha m} \tag{2.2}$$

for all m and all $x \in V_m \setminus V_0$, where

$$\Delta_m f(x) = \sum_{y \sim_m x} (f(y) - f(x)) \tag{2.3}$$

(there are 4 summands in each sum).

(c) For $0 < \alpha \leqslant \alpha_3 = \alpha_1 + \alpha_2$ define $\Lambda_\alpha^{(3)}(SG)$ as before with (2.1) replaced by

$$|\Delta_m f(x) - \Delta_m f(y)| \leqslant M\left(\tfrac{3}{5}\right)^{\alpha m} \tag{2.4}$$

for all m and all $x, y \in V_m \setminus V_0$ with $x \sim_m y$.

Remark. $\left(\frac{3}{5}\right)^{\alpha_2} = \frac{1}{5}$.

Lemma 2.2. *The space $\Lambda_\alpha^{(1)}(SG)$ may be identified with the Hölder space of functions satisfying*

$$|f(x) - f(y)| \leqslant M d(x,y)^\alpha \quad \textit{for all } x, y \in SG \tag{2.5}$$

for the resistance metric, or equivalently

$$|f(x) - f(y)| \leqslant M|x-y|^{\alpha/\alpha_1} \tag{2.6}$$

for the Euclidean distance.

Proof. It is clear that (2.1) is a special case of (2.5). Now assume (2.1) holds. Consider a cell $F_w(SG)$ of order m. If $x \in F_w(SG)$ and z is one of the boundary points of the cell, it is easy to see that there exists a *telescoping ladder* $(x_m, x_{m+1}, \dots)$ joining them, where $x_m = z$ and $x = \lim_{k\to\infty} x_k$ with $x_k \in V_k$ and $x_{k-1} \sim_k x_k$ for all $k \geqslant m+1$. By summing estimates (2.1) along the rungs of the ladder we obtain a convergent geometric series, hence

$$|f(x) - f(z)| \leqslant c_\alpha M\left(\tfrac{3}{5}\right)^{m\alpha}. \tag{2.7}$$

Now if $d(x,y) < \frac{1}{6}\left(\frac{3}{5}\right)^m$, then x and y belong either to the same or adjacent cells of order m. (The reason is that otherwise we can define a piecewise harmonic spline φ by taking φ on V_m to be zero except for the 3 boundary points of the cell containing x, so that $\varphi(x) = 1$ and $\varphi(y) = 0$; but there is an upper bound of $6 \cdot (5/3)^m$ for $E(\varphi, \varphi)$, giving a lower bound of $\frac{1}{6}\left(\frac{3}{5}\right)^m$ for $d(x,y)$.) We then choose telescoping ladders from x and y to the same z, obtaining $|f(x) - f(y)| \leqslant 2c_\alpha M\left(\frac{3}{5}\right)^{m\alpha}$, and the optimal choice of m yields (2.5). Since $d(x,y)$ is equivalent to $|x-y|^{1/\alpha_1}$ (the estimate $|x-y|^{1/\alpha_1} \leqslant c d(x,y)$ was given above, and the reverse estimate is also simple) we also have (2.6). $\square$

Note that it is not really necessary to assume f is continuous on SG. If f is defined on V_* and satisfies (2.1) then the proof of the lemma shows that f is uniformly continuous on V_* hence extends to a continuous function on SG. Also, it suffices to have (2.1) hold only on $V_m \setminus V_0$, since we can pick up the boundary points in the limit. Note that the Whitney extension theorem implies that $\Lambda_\alpha^{(1)}(SG)$ is equal to the space of restrictions to SG of $\mathrm{Lip}(\alpha/\alpha_1)$ functions on the plane. This explains why the condition $\alpha \leqslant \alpha_1$ is natural in the definition. It is easy to show that only the constants satisfy (2.1) for $\alpha > \alpha_1$. Nevertheless, we will see that $\Lambda_\alpha^{(1)}(SG)$ should be identified as $\Lambda_\alpha(SG)$ only for $\alpha < 1$.

Lemma 2.3. *$\Lambda_\alpha^{(1)}(SG) = \Lambda_\alpha^{(2)}(SG)$ for $0 < \alpha < 1$.*

Proof. It is clear by summing that (2.1) implies (2.2), the constant increasing by a factor of 4, regardless of the value of α. The reverse estimate is more subtle, with a factor that blows up as $\alpha \to 1^-$. It is proved by an argument analogous to the original argument of Zygmund on the line.

The key to the argument is a combinatorial identity that expresses differences $f(x) - f(y)$ for $x \sim_m y$ in terms of differences $f(x') - f(y')$ for $x' \sim_{(m-1)} y'$ and terms $\Delta_m f(z)$. There are two cases, depending on whether or not one of the points x, y belongs to V_{m-1}. Fig. 1(a) shows the cell of level $m-1$ containing x and y in the first case, when neither x nor y is in V_{m-1}, and Fig. 1(b) shows the second case when $y \in V_{m-1}$ but $x \notin V_{m-1}$ (it is not possible to have both x and y in V_{m-1} since $x \sim_m y$). In case (a) the identity is

$$(f(x) - f(y)) = \tfrac{1}{5}(f(x') - f(y')) - \tfrac{1}{5}\Delta_m f(x) + \tfrac{1}{5}\Delta_m f(y). \tag{2.8}$$

In case (b) the identity is

$$f(x) - f(y)) = \tfrac{2}{5}(f(w') - f(y)) + \tfrac{1}{5}(f(z') - f(y)) - \tfrac{3}{10}\Delta_m f(x) - \tfrac{1}{10}\Delta_m f(z) - \tfrac{1}{10}\Delta_m f(w). \tag{2.9}$$

If we write

$$\delta_m = \sup\{|f(x) - f(y)| : x \sim_m y\} \tag{2.10}$$

and

$$\varepsilon_m = \sup\{|\Delta_m f(x)| : x \in V_m \backslash V_0\} \tag{2.11}$$

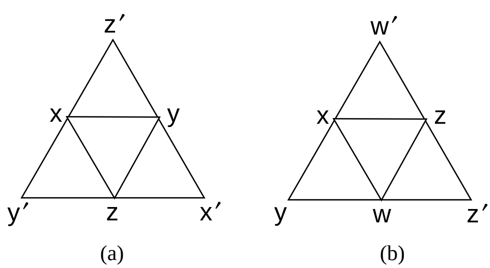

Fig. 1. The cell of level $m-1$ containing x and y in the two cases (a) $x \notin V_{m-1}$ and $y \notin V_{m-1}$, (b) $x \notin V_{m-1}$ and $y \in V_{m-1}$.

then we obtain the estimate

$$\delta_m \leqslant \tfrac{3}{5}\delta_{m-1} + \tfrac{1}{2}\varepsilon_m. \tag{2.12}$$

Since $\varepsilon_m \leqslant M(\frac{3}{5})^{m\alpha}$ by (2.2) we obtain by induction

$$\delta_m \leqslant \left(\frac{3}{5}\right)^m \delta_1 + \frac{1}{2} M \sum_{k=0}^{m} \left(\frac{3}{5}\right)^{(m-k)\alpha+k}.$$

The condition $\alpha < 1$ yields both

$$(\tfrac{3}{5})^m \leqslant (\tfrac{3}{5})^{m\alpha}$$

and

$$\sum_{k=0}^{m} \left(\frac{3}{5}\right)^{(m-k)\alpha+k} \leqslant c_\alpha \left(\frac{3}{5}\right)^{m\alpha}$$

and we can bound δ_1 by $2||f||_\infty$. Thus δ_m is bounded by a multiple of $(\frac{3}{5})^{m\alpha}$, yielding (2.1). $\square$

As in the case of the line, at the endpoint $\alpha = 1$ the above argument shows that (2.2) implies

$$|f(x) - f(x)| \leqslant Md(x,y)(1 + |\log d(x,y)|).$$

It is easy to give an example of a function in $\Lambda_\alpha^{(2)}$ but not in $\Lambda_\alpha^{(1)}$ for $\alpha > 1$, simply by taking any nonconstant harmonic function. We then have $\Delta_m f(x) = 0$, so (2.2) holds for any α, but at any vertex x where the normal derivative is nonzero we have $f(x) - f(y) \approx (\frac{3}{5})^m$ exactly. It seems likely that $\Lambda_1^{(2)}$ contains functions not in $\Lambda_1^{(1)}$, but to actually show this would require a more complicated example.

Lemma 2.4. (a) *$\Lambda_\alpha^{(3)}(SG) = \Lambda_\alpha^{(2)}(SG)$ for $0 < \alpha < \alpha_2$.* (b) *For $\alpha_2 < \alpha < \alpha_1 + \alpha_2$, $f \in \Lambda_\alpha^{(3)}(SG)$ if and only if f is in dom(Δ) and $\Delta f \in \Lambda_{\alpha-\alpha_2}^{(1)}(SG)$.*

Proof. (a) If (2.2) holds then so does (2.4), regardless of α, so we consider the converse. Consider first the case when $x \in V_m \backslash V_0$ also satisfies $x \in V_{m-1}$. Then Fig. 2 shows the two adjacent cells of level $m-1$ containing x. In this case we have the identity

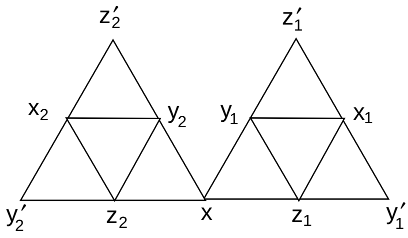

Fig. 2. Two adjacent cells of level $m-1$ containing the common vertex x.

$$\begin{aligned}\Delta_m f(x) = &\tfrac{1}{5}\Delta_{m-1} f(x) + \tfrac{1}{6}(\Delta_m f(x) - \Delta_m f(y_1)) \\ &+ \tfrac{1}{6}(\Delta_m f(x) - \Delta_m f(z_1)) + \tfrac{1}{6}(\Delta_m f(x) - \Delta_m f(y_2)) \\ &+ \tfrac{1}{6}(\Delta_m f(x) - \Delta_m f(z_2)) + \tfrac{1}{30}(\Delta_m f(y_1) - \Delta_m f(x_1)) \\ &+ \tfrac{1}{30}(\Delta_m f(z_1) - \Delta_m f(x_1)) + \tfrac{1}{30}(\Delta_m f(y_2) - \Delta_m f(x_2)) \\ &+ \tfrac{1}{30}(\Delta_m f(z_2) - \Delta_m f(x_2)). \end{aligned} \tag{2.13}$$

On the other hand, if say $x' \in V_m \backslash V_0$ but $x' \notin V_{m-1}$ you can always find $x \in V_{m-1} \backslash V_0$ such that $x \sim_m x'$. Then

$$\Delta_m f(x') = (\Delta_m f(x') - \Delta_m f(x)) + \Delta_m f(x)$$

and so

$$\begin{aligned}\Delta_m f(x') = &\tfrac{1}{5}\Delta_{m-1} f(x) + (\Delta_m f(x') - \Delta_m f(x)) \\ &+ \text{the other terms from (2.13).}\end{aligned} \tag{2.14}$$

So if we define ε_m by (2.11) and

$$\eta_m = \sup\{|\Delta_m f(x) - \Delta_m f(y)| : x \sim_m y, \ x, y \in V_m \backslash V_0\} \tag{2.15}$$

then from (2.13) and (2.14) we obtain the estimate

$$\varepsilon_m \leqslant \tfrac{1}{5}\varepsilon_{m-1} + \tfrac{9}{5}\eta_m. \tag{2.16}$$

The rest of the proof is the same as the proof of Lemma 2.3, but because of the factor $1/5$ in (2.16) as opposed to $3/5$ in (2.12), we have the restriction $\alpha < \alpha_2$.

(b) We begin with the easy direction, assuming $u \in \operatorname{dom} \Delta$ and $\Delta u \in \Lambda^{(1)}_{\alpha - \alpha_2}(SG)$. We use the formula

$$\Delta_m u(x) = \left(\frac{3}{5}\right)^m \int (\Delta u)\psi_x^{(m)}\, d\mu, \tag{2.17}$$

where $\psi_x^{(m)}$ is the piecewise harmonic spline of level m assuming the value 1 at x and 0 elsewhere on V_m. Note that if $x\sim_m y$ then $d(x',y')\leqslant c(3/5)^m$ if $x'\in\operatorname{supp}\psi_x^{(m)}$ and $y'\in\operatorname{supp}\psi_y^{(m)}$. Thus

$$|\Delta u(x')-\Delta u(y')|\leqslant cM(3/5)^{m(\alpha-\alpha_2)}=cM(3/5)^{m\alpha}5^m$$

by the assumption $\Delta u\in\Lambda^{(1)}_{\alpha-\alpha_2}(SG)$. From (2.17) we obtain

$$|\Delta_m u(x)-\Delta_m u(y)|\leqslant cM(\tfrac{3}{5})^{m\alpha}$$

since the measure of $\operatorname{supp}\psi_x^{(m)}$ is $2\cdot 3^{-m}$, as desired.

Conversely, assume $u\in\Lambda^{(3)}_\alpha(SG)$ and fix a point $x\in V_m\backslash V_0$. Then $x\in V_{m+k}$ for every $k\geqslant 1$ and we can use (2.13) to estimate

$$|5^{m+k}\Delta_{m+k}u(x)-5^{m+k-1}\Delta_{m+k-1}u(x)|\leqslant\tfrac{4}{5}5^{m+k}\eta_{m+k}. \tag{2.18}$$

The condition $\alpha>\alpha_2$ makes the right-hand side of (2.18) go to zero at a geometric rate, which implies that Δu exists (the estimates are uniform) and so

$$|\tfrac{2}{3}5^m\Delta_m u(x)-\Delta u(x)|\leqslant cM(\tfrac{3}{5})^{m(\alpha-\alpha_2)}. \tag{2.19}$$

If $y\sim_m x$ then using (2.19) for y in place of x and

$$|5^m\Delta_m u(x)-5^m\Delta_m u(y)|\leqslant cM5^m(\tfrac{5}{3})^{m\alpha}=cM(\tfrac{5}{3})^{m(\alpha-\alpha_2)},$$

we obtain $|\Delta u(x)-\Delta u(y)|\leqslant cM(\tfrac{5}{3})^{m(\alpha-\alpha_2)}$ as desired, with the constant depending on α and blowing up as $\alpha\to\alpha_2^+$. The fact that Δu is also bounded follows easily from the above estimates. □

Note that the restriction $\alpha\leqslant\alpha_1+\alpha_2$ in the definition of $\Lambda^{(3)}_\alpha$ is natural, for otherwise $\Lambda^{(1)}_{\alpha-\alpha_2}$ would just consist of constants, and $\Lambda^{(3)}_\alpha$ would just be the four-dimensional space of solutions of $\Delta u=c$. For a similar reason we need the condition $\alpha\leqslant\alpha_2$ in the definition of $\Lambda^{(2)}_\alpha$, for otherwise condition (2.2) would imply $\Delta u=0$, so $\Lambda^{(2)}_\alpha$ would reduce to the three-dimensional space of harmonic functions.

Definition 2.5. Let $\alpha>0$ and let k be the largest integer such that $k\alpha_2<\alpha$ (so $0<\alpha-k\alpha_2\leqslant\alpha_2$). Define $\Lambda_\alpha(SG)$ to be the space of bounded continuous functions u such that (if $k>0$) $u\in\operatorname{dom}(\Delta^k)$ and $\Delta^k u\in\Lambda^{(3)}_{\alpha-k\alpha_2}(SG)$, with corresponding norm.

Remark. If $\alpha-k\alpha_2<\alpha_2$ we can replace $\Lambda^{(3)}_{\alpha-k\alpha_2}$ by $\Lambda^{(2)}_{\alpha-k\alpha_2}$, while if $\alpha-k\alpha_2<1$ we can replace it by $\Lambda^{(1)}_{\alpha-k\alpha_2}$. It is clear from the definition that $u\in\Lambda_{\alpha+\alpha_2}(SG)$ if and only if $\Delta_u\in\Lambda_\alpha(SG)$, so the Laplacian acts as an "elliptic operator of order α_2" on the Λ_α scale of spaces. Also, it is clear that $\Lambda_\alpha\subseteq\Lambda_\beta$ if $\alpha>\beta$.

Definition 2.6. Let w be a word of length $|w| = n$, and let $F_w q_j = x$ be a vertex of the cell $F_w(SG)$. We say that u has a *normal derivative* at x with respect to the cell if

$$\partial_n u(x) = \lim_{m\to\infty} (\tfrac{5}{3})^{m+n}(2u(x) - u(F_w F_j^m q_{j+1}) - u(F_w F_j^m q_{j-1})) \tag{2.20}$$

exists. Note that $F_w F_j^m q_{j+1}$ and $F_w F_j^m q_{j-1}$ are the two neighbors of x in the cell $F_w F_j^m(SG)$. If x is not a boundary point of SG, then there are two cells of level n that have x as a boundary point of those cells, although our notation is deliberately vague in not indicating which cell is involved. We say that u satisfies the *matching condition* at such a point x if the sum of the two normal derivatives vanishes. Note that the matching condition says exactly

$$\lim_{m\to\infty} (\tfrac{5}{3})^m \Delta_m u(x) = 0, \tag{2.21}$$

but (2.21) alone does not imply the existence of either limit in (2.20).

Theorem 2.7. *If* $u \in \Lambda_\alpha(SG)$ for $\alpha > 1$, *then u has normal derivatives at all vertex points and the matching condition holds at all nonboundary points.*

Proof. We may assume, without loss of generality, that $\alpha < \alpha_2$ so $\Lambda_\alpha = \Lambda_\alpha^{(2)}$. The matching condition is easy, as (2.21) is an obvious consequence of the $\Lambda_\alpha^{(2)}$ condition. We can estimate the difference of successive terms on the right-hand side of (2.20) by using the identity

$$\begin{aligned}
&(2u(x) - u(F_w F_j^{m-1} q_{j+1}) - u(F_w F_j^{m-1} q_{j-1})) \\
&\quad - \tfrac{5}{3}(2u(x) - u(F_w F_j^m q_{j+1}) - u(F_w F_j^m q_{j-1})) \\
&\quad = -\tfrac{1}{3}(2\Delta_{m+n} u(F_w F_j^m q_{j+1}) + 2\Delta_{m+n} u(F_w F_j^m q_{j-1}) \\
&\qquad + \Delta_{m+n} u(F_w F_j^{m-1} F_{j+1} q_{j-1})).
\end{aligned} \tag{2.22}$$

Indeed, multiplying (2.22) by $(5/3)^{m+n-1}$ and using the $\Lambda_\alpha^{(2)}$ estimate we obtain

$$\begin{aligned}
&|(\tfrac{5}{3})^{m+n-1}(2u(x) - u(F_w F_j^{m-1} q_{j+1}) - u(F_w F_j^{m-1} q_{j-1})) \\
&\quad - (\tfrac{5}{3})^{m+n}(2u(x) - u(F_w F_j^m q_{j+1}) - u(F_w F_j^m q_{j-1}))| \\
&\quad \leqslant M(\tfrac{3}{5})^{(m+n)(\alpha-1)},
\end{aligned}$$

which implies the existence of the limit. $\square$

Theorem 2.8. $\Lambda_\alpha(SG)$ *forms an algebra under pointwise multiplication if* $\alpha < 2$. *However, this fails if* $\alpha > \alpha_2$ (*in fact* $f \cdot g \notin \Lambda_\alpha(SG)$ *if* $f, g \in \Lambda_\alpha(SG)$ *are nonconstant in this range*).

Proof. The negative result follows from the corresponding negative result for dom Δ proved in [BST]. The positive result is trivial for $\alpha < 1$ since $\Lambda_\alpha(SG) = \Lambda_\alpha^{(1)}(SG)$ and Hölder spaces always allow multiplication. In the range $\alpha < 2$ we have $\Lambda_\alpha = \Lambda_\alpha^{(2)}$, so we need to estimate $\Delta_m(fg)$ if $f, g \in \Lambda_\alpha^{(2)}$. We use the identity

$$\Delta_m(fg)(x) = f(x)\Delta_m g(x) + g(x)\Delta_m f(x) + \sum_{y \sim_m x} (f(x) - f(y))(g(x) - g(y)). \tag{2.23}$$

The first two terms are easily controlled because f and g are bounded and $\Delta_m f$ and $\Delta_m g$ are $O((\frac{3}{5})^{m\alpha})$. For the remaining term, note that there are only 4 summands, and $f, g \in \Lambda_{\alpha/2} = \Lambda_{\alpha/2}^{(1)}$ because $\alpha/2 < 1$, so both $f(x) - f(y)$ and $g(x) - g(y)$ are $O((\frac{3}{5})^{m\alpha/2})$. $\square$

Remark. It seems likely that the positive result holds when $\alpha = 2$, since the same proof will work if we have $\Lambda_2 \subseteq \Lambda_1^{(1)}$. In fact it seems plausible that $\Lambda_\alpha \subseteq \Lambda_1^{(1)}$ for any $\alpha > 1$, but on the other hand Λ_α is not contained in $\Lambda_\beta^{(1)}$ for $1 < \beta < \alpha$, since Λ_α contains harmonic functions, which are not in $\Lambda_\beta^{(1)}$. It is not at all clear what happens in the range $2 < \alpha \leqslant \alpha_2$.

Theorem 2.9. *Suppose* $0 < s_0 < s_1 < \alpha_2 + 1$. *Then the complex interpolation space* $[\Lambda_{s_0}(SG), \Lambda_{s_1}(SG)]_\theta$ *is contained in* $\Lambda_s(SG)$ *for* $0 < \theta < 1$ *where* $s = (1-\theta)s_0 + \theta s_1$.

Proof. For the indicated range of s values we may use the $\Lambda_\alpha^{(3)}(SG)$ characterization. If $F(z)$ is a bounded analytic function from the strip $0 < \operatorname{Re} z < 1$ into $\Lambda_{s_0}^{(3)}(SG) + \Lambda_{s_1}^{(3)}(SG)$, continuous up to the boundary and satisfying

$$||F(j + it)||_{\Lambda_{s_j}^{(3)}(SG)} \leqslant M \quad \text{for } j = 0, 1$$

and all real t, then it follows easily that $||F(\theta)||_{\Lambda_s^{(3)}(SG)} \leqslant M$. $\square$

Conjecture 2.10. (a) $[\Lambda_{s_0}(SG), \Lambda_{s_1}(SG)]_\theta \subseteq \Lambda_s(SG)$ *for any* $s_0, s_1 > 0$. (b) *the containment in* (a) *is in fact equality.*

Note that for (a) the same proof will work if we can find a single difference quotient characterization of $\Lambda_\alpha(SG)$ over arbitrarily large intervals. It seems plausible that the condition

$$|\Delta_m^k f(x)| \leqslant M(\tfrac{3}{5})^{\alpha m}$$

will work for $\alpha < k\alpha_2$.

3. Sobolev spaces on SG

We want to define spaces on SG that are the analog of the Sobolev spaces $L^p_s(\mathbb{R}^n)$ for $s \geqslant 0$, $1 < p < \infty$, in terms of "pseudodifferential operators" that are analogs of the Riesz and Bessel potentials [St]. Every function $f \in L^2(SG)$ has two distinct eigenfunction expansions

$$\begin{cases} f = \sum_{j=1}^{\infty} \hat{f}_{j,D}\varphi_{j,D}, \\ f = \sum_{j=1}^{\infty} \hat{f}_{j,N}\varphi_{j,N}, \end{cases} \tag{3.1}$$

where $\{\varphi_{j,D}\}$ and $\{\varphi_{j,N}\}$ are an orthonormal basis of Dirichlet and Neumann eigenfunctions of the Laplacian,

$$\begin{cases} -\Delta\varphi_{j,D} = \lambda_{j,N}\varphi_{j,N}, \\ -\Delta\varphi_{j,N} = \lambda_{j,N}\varphi_{j,N}. \end{cases} \tag{3.2}$$

(We will drop the subscripts D and N when we make statements that hold for both expressions.) We have $\lambda_{1,N} = 0$, corresponding to $\varphi_{1,N} \equiv 1$, but all the other eigenvalues are positive (we assume they are arranged in nondecreasing order).

Given any bounded function $F : \mathbb{R}_+ \to \mathbb{C}$ we can define spectral multiplier operators $F(-\Delta_D)$ and $F(-\Delta_N)$ by

$$F(-\Delta)f = \sum_{j=1}^{\infty} F(\lambda_j)\hat{f}_j\varphi_j. \tag{3.3}$$

Of particular importance are the heat operator

$$e^{t\Delta}f = \sum_{j=1}^{\infty} e^{-t\lambda_j}\hat{f}_j\varphi_j, \quad t > 0, \tag{3.4}$$

the Bessel potentials

$$(I - \Delta)^{-s}f = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-s}\hat{f}_j\varphi_j, \ \operatorname{Re} s \geqslant 0, \tag{3.5}$$

and in the Dirichlet case the Riesz potentials

$$(-\Delta_D)^{-s}f = \sum_{j=1}^{\infty} (\lambda_{j,D})^{-s}\hat{f}_{j,D}\varphi_{j,D}, \ \operatorname{Re} s \geqslant 0. \tag{3.6}$$

(In the Neumann case we could also define Riesz potentials by factoring out the constants.) We will need to use some known information about these operators that have been obtained by quite different methods. First, we state the heat kernel estimates originally due to Barlow and Perkins [BP] (see also [HK]).

Proposition 3.1. *The heat operator is given by integration with respect to a positive heat kernel* $p_t(x,y)$,

$$e^{-t\Delta}f(x) = \int p_t(x,y)f(y)\,d\mu(y). \tag{3.7}$$

In the Neumann case, $p_t(x,y)$ *is bounded above and below by constant multiples of*

$$t^{-d/\alpha_2}\exp\left(-c\left(\frac{d(x,y)^{\alpha_2}}{t}\right)^{1/d}\right), \quad \textit{for } 0<t<1. \tag{3.8}$$

The constant c in (3.8) *may be different in the upper and lower estimate, and the upper estimate holds in the Dirichlet case. The same type of estimates hold for* $t^k(\frac{\partial}{\partial t})^k p_t(x,y)$ *for any integer k.*

Using the above heat kernel estimates and the spectral multiplier results of Hebisch [He] and Duong et al. [DOS] we obtain the following estimates for the Riesz and Bessel potentials, or more generally any F satisfying standard Hörmander multipler estimates. Although this application is not explicitly stated in either paper, I have been informed by the authors that they were well aware of this.

Proposition 3.2. *For* $\operatorname{Re} s \geqslant 0$, *the Riesz and Bessel potentials are bounded on* L^p, $1<p<\infty$, *with operator norm of at most polynomial growth in* $\operatorname{Im} s$ (*when* $\operatorname{Re} s = 0$) *for fixed p.*

This result for $\operatorname{Re} s>0$ is more elementary, as we will see later.

Definition 3.3. We define the spaces $L^p_{s,D}(SG)$ and $L^p_{s,N}(SG)$ for $s>0$ and $1<p<\infty$ to be the images of L^p under $(I-\Delta_D)^{-s/\alpha_2}$ and $(I-\Delta_N)^{-s/\alpha_2}$, respectively, with the norm of $(I-\Delta_D)^{-s/\alpha_2}f$ in $L^p_{s,D}(SG)$ given by $\|f\|_p$, and the same in the Neumann case. Note that by Proposition 3.2 we may regard $L^p_{s,D}(SG)$ and $L^p_{s,N}(SG)$ as closed subspaces of L^p, with

$$\|u\|_p \leqslant c\|u\|_{L^p_{s,D}}.$$

Also, in the Dirichlet case, we may use Riesz potentials in place of Bessel potentials.

Because of the boundary conditions, the two types of spaces are not identical. However, because the boundary is finite, they differ by a finite dimensional space, which depends on s.

Corollary 3.4. *If* $0<s_0\leqslant s_1<\infty$ *and* $1<p_0,p_1<\infty$ *then the complex interpolation space* $[L^{p_0}_{s_0,D}(SG), L^{p_1}_{s_1,D}(SG)]_\theta$ *may be identified with* $L^p_{s,D}(SG)$ *where* $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$, *and* $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$. *The same is true in the Neumann case.*

Proof. The proof is the same as for the Euclidean space case, using Proposition 3.2. $\square$

Lemma 3.5. $L^p_{\alpha_2,D}(SG)$ *is the subspace of* $dom_{L^p}(\Delta)$ *of functions satisfying Dirichlet boundary conditions*; *it has codimension* 3, *with the space of harmonic functions serving as a complementary subspace.* $L^p_{\alpha_2,N}(SG)$ *is the subspace of* $dom_{L^p}(\Delta)$ *of functions satisfying Neumann boundary condition*; *it has codimension* 2, *with the space of harmonic functions modulo constants serving as a complementary subspace.*

Proof. Suppose $u \in dom_{L^p}\,\Delta$, meaning $u \in dom\,\mathscr{E}$ and there exists $f \in L^p$, identified as $-\Delta u$, such that

$$\int fv\,d\mu = \mathscr{E}(u,v) \quad \text{for all } v \in \mathrm{dom}_0\,\mathscr{E}. \tag{3.9}$$

In particular, if we take $v = \varphi_{j,D}$, then $\hat{f}_{j,D} = \mathscr{E}(u,\varphi_{j,D})$. If u also satisfies Dirichlet boundary conditions, then $\mathscr{E}(u,\varphi_{j,D}) = \lambda_{j,D}\int u\varphi_{j,D}\,d\mu$ hence

$$\hat{f}_{j,D} = \lambda_{j,D}\hat{u}_{j,D}. \tag{3.10}$$

This implies $u = (I - \Delta_D)^{-1}(f+u)$ hence $u \in L^p_{\alpha_2,D}(SG)$.

Conversely, suppose $u \in L^p_{\alpha_2,D}(SG)$, so $u = (I-\Delta_D)^{-1}g$ for some $g \in L^p$. Let $f = g - u$. Then (3.9) holds for any v represented by a finite sum in the Dirichlet eigenfunction expansion (3.1). But for any $v \in dom_0\,\mathscr{E}$, $v - v_m$ converges to zero in energy as $m \to \infty$, where $v_m = \sum_{j=1}^m \hat{v}_{j,D}\varphi_{j,D}$. The vanishing on the boundary of v means that $v - v_m$ converges to zero uniformly, hence we can pass to the limit from $\int fv_m\,d\mu = \mathscr{E}(u,v_m)$ to obtain (3.9), so $f \in dom_{L^p}\,\Delta$. A similar argument shows that the Dirichlet expansion (3.1) for u converges uniformly, so u satisfies Dirichlet boundary conditions. Of course, the harmonic functions also belong to $dom_{L^p}\,\Delta$, and by subtracting off a suitable harmonic function h for any u in $dom_{L^p}\,\Delta$ we can make $u - h$ satisfy Dirichlet boundary conditions. Moreover, any nonzero harmonic function fails to satisfy the Dirichlet boundary conditions, so it is not in $L^p_{\alpha_2,D}(SG)$. This completes the proof that $dom_{L^p}\,\Delta = L^p_{\alpha_2,D}SG) + \mathscr{H}_0$ with no overlap, where $\mathscr{H}_0$ denotes the space of harmonic functions.

The argument for the Neumann case is similar, but first we need an extension of (3.9) that holds for all $v \in dom\,\mathscr{E}$: if $u \in dom_{L^p}\,\Delta$ then normal derivatives $\partial_n u$ exist, and there exists $f \in L^p$ such that

$$\int fv\,d\mu = \mathscr{E}(u,v) - \sum_{\partial SG} v\partial_n u. \tag{3.11}$$

This is proved in [Ki] for $p = 2$, but the extension to all p is straightforward. In particular, we get back to (3.9) if we assume u satisfies Neumann boundary conditions. The argument in the first direction then proceeds as before. In the converse direction we need the additional observation that $\int f\,d\mu = \int f\varphi_{1,N}\,d\mu = \mathscr{E}(f,1) = 0$. Thus we are free to modify v_m by an additive constant in the identity $\int f v_m\,d\mu = \mathscr{E}(u,v_m)$, and the additive constant allows us to pass from $v - v_m \to 0$ in energy to $v - v_m \to 0$ uniformly. Thus we obtain as before that $u \in L^p_{\alpha_2,N}(SG)$ implies $u \in dom_{L^p}\,\Delta$. To prove that u satisfies Neumann boundary conditions we note first that (3.11) implies that $\partial_n u$ exists, and

$$\int f\varphi_{j,N}\,d\mu = \mathscr{E}(u,\varphi_{j,N}) - \sum_{\partial SG} \varphi_{j,N}\partial_n u.$$

However, $\int f\varphi_{,N}\,d\mu = \mathscr{E}(u,\varphi_{j,N})$ by the construction of f, so $\partial_n u$ must vanish identically on the boundary.

Finally, we claim $dom_{L^p}\,\Delta = L^p_{\alpha_2,N}(SG) + \mathscr{H}_0$ with a one-dimensional overlap, namely the constants. Given any $u \in dom_{L^p}\,\Delta$, we may subtract a constant to make $\int u\,d\mu = 0$. Then (3.11) with $v \equiv 1$ implies $\sum_{\partial SG} \partial_n u = 0$. But the normal derivatives of harmonic functions span the two-dimensional space given by this condition, so we can find a harmonic function h such that $u - h$ satisfies the Neumann boundary conditions, hence $u - h \in L^p_{\alpha_2,N}(SG)$. $\square$

Corollary 3.6. $L^p_{s,D}(SG) + \mathscr{H}_0 = L^p_{s,N}(SG) + \mathscr{H}_0$ *for* $0 \leqslant s \leqslant \alpha_2$.

Proof. This is true for $s = \alpha_2$ by the lemma, and it is trivially true for $s = 0$ because $\mathscr{H}_0 \subseteq L^p = L^p_{0,D} = L^p_{0,N}$. Interpolate. $\square$

Presumably, for s small enough (depending on p) $\mathscr{H}_0 \subseteq L^p_{s,D}(SG)$ and similarly for the Neumann case. In principle, for $p = 2$ one could decide the exact range of s values by computing expansions (3.1) for harmonic functions, or at least getting sharp estimates on the coefficients. As part (d) of the next theorem shows, $\mathscr{H}_0 \subseteq L^2_{\alpha_2/2,N}(SG)$ but $\mathscr{H}_0$ is not contained in $L^2_{\alpha_2/2,D}(SG)$.

Theorem 3.7. (a) $L^p_{k\alpha_2,D}(SG)$ *coincides with the space of functions* $u \in dom_{L^p}\,\Delta^k$ *such that* $\Delta^j u$ *satisfies Dirichlet boundary conditions for* $j < k$, *and similarly for the Neumann case.*

(b) $dom_{L^p}\,\Delta^k = L^p_{k\alpha_2,D}(SG) + \mathscr{H}_{k-1} = L^p_{k\alpha_2,N}(SG) + \mathscr{H}_{k-1}$, *where* $\mathscr{H}_j$ *denotes the* $3(j+1)$*-dimensional space of solutions of* $\Delta^{j+1}u = 0$.

(c) $L^p_{s,D}(SG) + \mathscr{H}_{k-1} = L^p_{s,N}(SG) + \mathscr{H}_{k-1}$ *for* $0 \leqslant s \leqslant k\alpha_2$.

(d) $L^2_{\alpha_2/2,D}(SG) = dom_0\,\mathscr{E}$ *and* $L^2_{\alpha_2/2,N}(SG) = dom\,\mathscr{E}$, *where* $dom_0\,\mathscr{E}$ *denotes the functions in* $dom\,\mathscr{E}$ *vanishing on the boundary.*

Proof. Conditions (a) and (b) follow from Lemma 3.5 by induction. Then (c) follows from (b) by interpolation as in Corollary 3.6.

(d) Consider expansion (3.1) for $f \in L^2_{\alpha_2/2,D}(SG)$ or $L^2_{\alpha_2/2,N}(SG)$. For simplicity of notation write either one as $f = \sum_{j=1}^{\infty} c_j \varphi_j$, and let $f_m = \sum_{j=1}^{m} c_j \varphi_j$. We know $\sum_{j=1}^{\infty} \lambda_j |c_j|^2 < \infty$. It follows that $\{f_m\}$ is a Cauchy sequence in energy, with $\mathscr{E}(f_m, f_m) = \sum \lambda_j |c_j|^2$. In the Dirichlet case, each f_m vanishes at the boundary, so f_m converges to f in energy and uniformly, so $f \in dom\, \mathscr{E}$ and f vanishes at the boundary. In the Neumann case $\int f_m \, d\mu = c_1$ for all m, so again f_m converges in energy and uniformly to f, so $f \in dom\, \mathscr{E}$.

Conversely, suppose $f \in dom\, \mathscr{E}$, and in the Dirichlet case vanishes at the boundary. Expansion (3.1) still exists in L^2, with coefficients given by $c_j = \int f \varphi_j \, d\mu$. We need to prove $\sum_{j=1}^{\infty} \lambda_j |c_j|^2 < \infty$. The key observation is that

$$\mathscr{E}(f, \varphi_j) = \lambda_j \int f \varphi_j \, d\mu.$$

In the Dirichlet case this (3.9), and in the Neumann case (3.11) (here $f = v$ and $\varphi_j = u$). Thus $\lambda_j c_j = \mathscr{E}(u, \varphi_j)$ and so

$$\sum_{j=1}^{m} \lambda_j |c_j|^2 = \mathscr{E}(f, f_m).$$

But a direct calculation shows

$$\sum_{j=1}^{m} \lambda_j |c_j|^2 = \mathscr{E}(f_m, f_m),$$

so

$$0 \leqslant \mathscr{E}(f - f_m, f - f_m) = \mathscr{E}(f, f) - \sum_{j=1}^{m} \lambda_j |c_j|^2.$$

This yields

$$\sum_{j=1}^{\infty} \lambda_j |c_j|^2 \leqslant \mathscr{E}(f, f). \qquad \square$$

Definition 3.8. The Sobolev space $L^p_s(SG)$ for $s > 0$ and $1 < p < \infty$ is defined to be $L^p_{s,D}(SG) + \mathscr{H}_{k-1} = L^p_{s,N}(SG) + \mathscr{H}_{k-1}$ where k is chosen to be the smallest integer such that $s \leqslant k\alpha_2$.

Theorem 3.9. (a) *The Sobolev spaces are stable under complex interpolation.*

(b) $u \in L^p_s(SG)$ *for* $s \geqslant \alpha_2$ *if and only if* $\Delta u \in L^p_{s-\alpha_2}(SG)$.

Proof. We observe that $\mathscr{H}_j \subseteq dom_{L^p}\, \Delta^k$ for any j and k, so in the definition of $L^p_s(SG)$ we are free to choose any k such that $s \leqslant k\alpha_2$, not just the smallest. Then (a) and (b) follow easily. $\square$

We are now in a position to study the analog of the Sobolev embedding theorems. We will see that Δ behaves like an elliptic pseudo-differential operator of order α_2 on a space of dimension d. When $s < d/p$, $L^p_s(SG)$ embeds in a Lebesgue space, whereas when $s > d/p$, $L^p_s(SG)$ embeds in a Hölder–Zygmund space.

Lemma 3.10. *The Bessel potential operator* $(I - \Delta_N)^{-s/\alpha_2}$ *is given by integration with respect to a positive kernel* $G_s(x, y)$,

$$(I - \Delta_N)^{-s/\alpha_2} f(x) = \int G_s(x, y) f(y)\, d\mu(y), \tag{3.12}$$

where G_s *satisfies the estimate*

$$G_s(x, y) \leqslant c_s d(x, y)^{s-d} \quad \text{for } s < d. \tag{3.13}$$

Proof. We have

$$(I - \Delta_N)^{-s/\alpha_2} f = c_s \int_0^\infty e^{t\Delta_N} f e^{-t} t^{\frac{s}{\alpha_2}-1}\, dt,$$

so (3.7) implies (3.12) with

$$G_s(x, y) = c_s \int_0^\infty p_{t,N}(x, y) e^{-t} t^{\frac{s}{\alpha_2}-1}\, dt. \tag{3.14}$$

Using estimate (3.8) for $t < 1$ and the trivial estimate that $p_{t,N}(x, y)$ is uniformly bounded for $t \geqslant 1$, we see that the integral in (3.14) converges for $x \neq y$, and G_s is positive since $p_{t,N}$ is positive. The contribution to the integral in (3.14) corresponding to $t \geqslant 1$ is bounded above by a constant multiple of $\int_1^\infty e^{-t} t^{\frac{s}{\alpha_2}-1}\, dt$ and hence is bounded, consistent with (3.13) where the right-hand side is bounded below. For $t < 1$ we can drop the e^{-t} factor and bound the integrand by

$$t^{\frac{(s-d)}{\alpha_2}-1} \exp\left(-c\left(\frac{d(x, y)^{\alpha_2}}{t}\right)^{1/d}\right).$$

We then extend the integral to $[0, \infty)$ using this bound, to get

$$\int_0^\infty t^{\frac{(s-d)}{\alpha_2}-1} \exp\left(-c\left(\frac{d(x, y)^{\alpha_2}}{t}\right)^{1/d}\right) dt = c_s d(x, y)^{s-d}$$

for $s < d$. $\square$

Theorem 3.11. *If $s < d/p$ then $L^p_s(SG) \subseteq L^q$ for $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.*

Proof. Since $\mathscr{H}_k \subseteq L^q$ trivially for all k and q, it suffices to show $L^p_{s,N}(SG) \subseteq L^q$, or equivalently $(I - \Delta_N)^{-s/\alpha_2}$ is a bounded operator from L^p to L^q. This is a consequence of the standard fractional integration theorem if we can establish the weak-type estimate

$$\mu\{x:\ G_s(x,y) \geqslant t\} \leqslant ct^{-\frac{d}{d-s}} \tag{3.15}$$

uniformly in y. But this is an immediate consequence of (3.13) and the measure/metric relationship

$$\mu\{x:\ d(x,y) \leqslant t\} \leqslant ct^d. \qquad \square \tag{3.16}$$

Remark. (1) It is also possible to prove the theorem from an $L^1 \to L^\infty$ estimate for the heat semigroup, using a "generic" argument [VSC]. The required estimate,

$$||e^{-t\Delta} f||_\infty \leqslant ct^{-d/\alpha_2} \quad \text{for } 0 < t \leqslant 1$$

is an immediate consequence of (3.8), but it is in fact a much weaker result (on-diagonal as opposed to off-diagonal heat kernel estimates).

(2) In the boundary case $s = d/p$ we can establish the analog of Trudinger's exponential integrability result [Tr], namely $\int \exp(a|f|^{p'})\, d\mu < \infty$ for a suitable constant a if $f \in L^p_{d/p}(SG)$. The proof given in [S2] is valid here because of (3.13).

Lemma 3.12. (a) $L^2_{\alpha_2/2}(SG) \subseteq \Lambda_{1/2}(SG)$. (b) $L^p_{k\alpha_2}(SG) \subseteq \Lambda_{k\alpha_2 - d/p}(SG)$ *for* $k = 1, 2, \ldots$.

Proof. (a) We have $L^2_{\alpha_2/2}(SG) = dom\, \mathscr{E}$, and $dom\, \mathscr{E} \subseteq \Lambda^{(1)}_{1/2}(SG)$ by the definition of the effective resistance metric.

(b) It suffices to show this for $k = 1$, because Δ acts on both scales of spaces with order α_2. Since $\mathscr{H}_0$ belongs to all Hölder–Zygmund spaces, it suffices to show that $L^p_{\alpha_2,D}(SG) \subseteq \Lambda^{(2)}_{\alpha_2 - d/p}(SG)$, or $(-\Delta_D)^{-1} f \in \Lambda^{(2)}_{\alpha_2 - d/p}(SG)$ for $f \in L^p$. We can do this because $(-\Delta_D)^{-1}$ is the Green's operator: $(-\Delta_D)^{-1} f = u$ means $-\Delta u = f$ and u vanishes on the boundary. Kigami [Ki] shows that

$$u(x) = \int G(x,y) f(y)\, d\mu(y) \tag{3.17}$$

for the Green's function G given explicitly by

$$G(x,y) = \sum_{m=0}^{\infty} \left(\frac{3}{5}\right)^m \sum_{|w|=m} \Psi_w(x,y), \tag{3.18}$$

where $\Psi_w(x,y)$ is supported in $F_w(SG) \times F_w(SG)$, and given by an explicit expression. For our purposes the important facts are that $G(x,y)$ as a function of x is harmonic away from $x = y$, and its Laplacian is minus the delta function at y.

For each $x \in V_m \backslash V_0$, let $U_m(x)$ denote the union of the two cells of level m containing x. We will use the formula, for any function $g \in dom\, \Delta$,

$$\Delta_m g(x) = \left(\frac{3}{5}\right)^m \int_{U_m(x)} \psi_x^{(m)} \Delta g(y)\, d\mu(y), \tag{3.19}$$

where $\psi_x^{(m)}$ is the piecewise harmonic spline on level m equal to 1 at x and 0 at all other vertices in V_m, which is a consequence of the Gauss–Green formula. Note that this implies

$$\Delta_m G(x,y) = 0 \quad \text{for } y \notin U_m(x), \tag{3.20}$$

while for $y \in U_m(x)$

$$|\Delta_m G(x,y)| \leqslant c\left(\tfrac{3}{5}\right)^m \tag{3.21}$$

(strictly speaking we need to apply (3.19) to $g(x) = \int G(x,z) f(z)\, d\mu(z)$ and let f approach a delta function at y). Then

$$\Delta_m u(x) = \int_{U_m(x)} \Delta_m G(x,y) f(y)\, d\mu(y)$$

by (3.18) and (3.20), and then

$$|\Delta_m u(x)| \leqslant c\left(\frac{3}{5}\right)^m \int_{U_m(x)} |f(y)|\, d\mu(y).$$

Hölder's inequality then yields

$$|\Delta_m u(x)| \leqslant c\left(\tfrac{3}{5}\right)^m \left(\tfrac{1}{3}\right)^{m/p'} ||f||_p$$

since $\mu(U_m(x)) = \frac{2}{3^m}$. But this says exactly that $u \in \Lambda^{(2)}_{\alpha_2 - d/p}(SG)$. $\square$

Theorem 3.13. *For* $s > d/p$, we have $L^p_s(SG) \subseteq \Lambda_{s-d/p}(SG)$ *under the following conditions*:

(a) $\alpha_2/2 < s \leqslant \alpha_2$ and $|\frac{1}{p} - \frac{1}{2}| < (\frac{s}{\alpha_2} - \frac{1}{2})$
(b) $s \geqslant \alpha_2$, *provided Conjecture* 2.10(a) *holds*.

Proof. (a) Interpolate between Lemmas 3.12(a) and (b) with $k = 1$, using Theorem 2.9.

(b) Interpolate between Lemma 3.12(b) with different values of k using Conjecture 2.10(a). $\square$

Conjecture 3.14. *The same result holds for all* $s > d/p$.

4. Besov spaces on SG

The Besov spaces on $\mathbb{R}^n$ are naturally thought of as generalizations of the Hölder–Zygmund spaces on $\mathbb{R}^n$, so in this section we effect a similar extension of the spaces $\Lambda_\alpha(SG)$ to $\Lambda^{p,q}_\alpha(SG)$, for $1 \leqslant p, q \leqslant \infty$, with the identification $\Lambda_\alpha(SG) = \Lambda^{\infty,\infty}_\alpha(SG)$. We will limit the discussion to the range $\alpha > d/p$ where the functions in $\Lambda^{p,q}_\alpha(SG)$ are automatically continuous, since our definitions only involve the values of the function on V_*, a countable set of points. To deal with $\alpha \leqslant d/p$ will require new ideas.

Definition 4.1. (a) For $\frac{d}{p} < \alpha \leqslant \frac{d}{p} + \alpha_1$ define $(\Lambda^{p,q}_\alpha)^{(1)}(SG)$ to be the Banach space of all bounded continuous functions on SG satisfying

$$\begin{cases} \left(\sum_{m=0}^{\infty} (\delta_{m,p} r^{-m\alpha})^q\right)^{1/q} \leqslant M & \text{if } q < \infty, \\ \sup_m \delta_{m,p} r^{-m\alpha} \leqslant M & \text{if } q = \infty \end{cases} \tag{4.1}$$

for $r = 3/5$ and

$$\delta_{m,p} = \begin{cases} \left(\frac{1}{3^m} \sum_{x \sim_m y} |f(x) - f(y)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup\{|f(x) - f(y)| : x \sim_m y\} & \text{if } p = \infty. \end{cases} \tag{4.2}$$

(b) For $\frac{d}{p} < \alpha < \frac{d}{p} + \alpha_2$ define $(\Lambda^{p,q}_\alpha)^{(2)}(SG)$ by (4.1) with $\delta_{m,p}$ replaced by $\varepsilon_{m,p}$ given by

$$\varepsilon_{m,p} = \begin{cases} \left(\frac{1}{3^m} \sum_{x \in V_m \setminus V_0} |\Delta_m f(x)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup\{|\Delta_m f(x)| : x \in V_m \setminus V_0\} & \text{if } p = \infty. \end{cases} \tag{4.3}$$

(c) For $\frac{d}{p} < \alpha \leqslant \frac{d}{p} + \alpha_3$ define $(\Lambda_\alpha^{p,q})^{(3)}(SG)$ by (4.1) with $\delta_{m,p}$ replaced by $\eta_{m,p}$ given by

$$\eta_{m,p} = \begin{cases} \left(\frac{1}{3^m}\sum_{x\sim_m y} |\Delta_m f(x) - \Delta_m f(y)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup\{|\Delta_m f(x) - \Delta_m f(y)| : x \sim_m y\} & \text{if } p = \infty. \end{cases} \tag{4.4}$$

Lemma 4.2. (a) $(\Lambda_\alpha^{p,q_1})^{(j)}(SG) \subseteq (\Lambda_\alpha^{p,q_2})^{(j)}(SG)$ *if* $q_1 < q_2$.

(b) $(\Lambda_\alpha^{p,q})^{(j)}(SG) \subseteq \Lambda_{\alpha-d/p}^{(j)}(SG)$.

(c) $(\Lambda_{\alpha_1}^{p_1,q})^{(j)}(SG) \subseteq (\Lambda_{\alpha_2}^{p_2,q})^{(j)}(SG)$ *if* $p_1 < p_2$ *and* $\alpha_1 - \frac{d}{p_1} = \alpha_2 - \frac{d}{p_2}$.

Proof. (a) This is just the containment $\ell^{q_1} \subseteq \ell^{q_2}$.

(b) If $p = \infty$ this follows from (a). If $p < \infty$ we use the trivial estimate that the sum in (4.2), (4.3) or (4.4) dominates an individual summand (and again for the sum in (4.1) if $q < \infty$). We obtain $|f(x) - f(y)| \leqslant M3^{m/p}r^{m\alpha}$ if $x \sim_m y$ in the $j = 1$ case. This is the desired result since $3^{1/p} = r^{-d/p}$. Similarly for $j = 2$ or 3.

(c) This is a straightforward application of Hölder's inequality. $\square$

This gives the automatic continuity of functions in these spaces if $\alpha > d/p$.

Lemma 4.3. $(\Lambda_\alpha^{p,q})^{(1)}(SG) = (\Lambda_\alpha^{p,q})^{(2)}(SG)$ *if*

$$\frac{d}{p} < \alpha < 1 + \frac{\log(\frac{3}{2+3^{-p}})}{p\log 5/3}. \tag{4.5}$$

Proof. Again it is clear that the $(\Lambda_\alpha^{p,q})^{(1)}$ estimate implies the $(\Lambda_\alpha^{p,q})^{(2)}$. The reverse direction follows the outline of the proof of Lemma 2.3, in particular the identities (2.8) and (2.9). We need to find the analog of (2.12) for $\delta_{m,p}$ and $\varepsilon_{m,p}$, where (2.12) will be the case $p = \infty$. The estimate will take the form

$$\delta_{m,p} \leqslant a_p\delta_{(m-1),p} + b_p\varepsilon_{m,p}, \tag{4.6}$$

where

$$a_p = \frac{3}{5}\left(\frac{2+3^{-p}}{3}\right)^{1/p} \tag{4.7}$$

and b_p is a constant whose value is irrelevant to what follows.

Indeed, each cell of level $m-1$ contains 3 edges of level $m-1$ and 9 edges of level m, and the latter may be sorted into 6 "exterior" edges for which (2.9) holds, and 3 "interior" edges for which (2.8) holds. When we add the estimates from each of the level $m-1$ cells we will obtain (4.6), so it suffices to show

$$\left(\frac{1}{3^m}\sum_{x\sim_m y}|f(x)-f(y)|^p\right)^{1/p}\leqslant a_p\left(\frac{1}{3^{m-1}}\sum_{x\sim_{(m-1)}y}|f(x)-f(y)|^p\right)^{1/p}+b_p\left(\frac{1}{3^m}\sum_{x\in V_m}|\Delta_m f(x)|^p\right)^{1/p}, \tag{4.8}$$

where the sums are restricted to the single level $m-1$ cell. To do this we use the sharp elementary estimate

$$|x+2y|^p+|2x+y|^p\leqslant 3^p(|x|^p+|y|^p) \tag{4.9}$$

to the six identities of the form (2.9). Note that each $m-1$ edge $x\sim_{(m-1)}y$ contributes a term $f(x)-f(y)$ on the right-hand side of (2.8) one time, and on the right-hand side of (2.9) four times, twice with a factor of $1/5$ and twice with a factor of $2/5$. This yields (4.8) with $a_p=\frac{1}{3^{1/p}}\cdot\frac{(1+2\cdot 3^p)^{1/p}}{5}$, which simplifies to (4.7).

From (4.6) we obtain

$$\left(\sum_{m=0}^{\infty}(\delta_{m,p}r^{-m\alpha})^q\right)^{1/q}\leqslant a_p r^{-\alpha}\left(\sum_{m=0}^{\infty}(\delta_{m,p}r^{-m\alpha})^q\right)^{1/q}+b_p\left(\sum_{m=0}^{\infty}(\varepsilon_{m,p}r^{-m\alpha})^q\right)^{1/q},$$

and this yields the desired estimate provided $a_p r^{-\alpha}<1$. In view of (4.7) this is exactly the right inequality in (4.5). $\square$

Remark. (1) Condition (4.5) implies the restrictions on α in Definition 4.1(a).

(2) For values of p close to 1, there are no values of α satisfying (4.5).

(3) For $p=2$ the condition is

$$1.0753301<\alpha<1.3439508.$$

Note that this excludes the value $\alpha_2/2$. This is significant because $(\Lambda^{2,\infty}_{\alpha_2/2})^{(1)}(SG)$ is just the space $dom\,\mathscr{E}$ of functions with finite energy. See [J,Kum] for other characterizations of this space. Presumably, there are functions in $(\Lambda^{2,\infty}_{\alpha_2/2})^{(2)}(SG)$ that do not have finite energy but we do not have any explicit example.

(4) For $\alpha>\alpha_2/2$, only constant functions belong to $(\Lambda^{2,\infty}_{\alpha})^{(1)}(SG)$, because such functions would have zero energy. This suggests that the upper bounds on α in Definition 4.1 are set too large, but we do not know what the correct upper bounds should be to make all the spaces infinite dimensional.

Lemma 4.4. *For* $d/p < \alpha < \alpha_2$, $(\Lambda_\alpha^{p,q})^{(3)}(SG) = (\Lambda_\alpha^{p,q})^{(2)}(SG)$.

Proof. The proof of the difficult half follows the outline of the proof of Lemma 2.4. In fact, we claim

$$\varepsilon_{m,p} \leqslant \tfrac{1}{5}\varepsilon_{m-1,p} + b_p \eta_{m,p}. \tag{4.10}$$

Note that estimates (2.13) and (2.14) both have the factor $1/5$ in front of the single $\Delta_{m-1}f$ term. Since there are 3 times as many summands in $\varepsilon_{m,p}$ than $\varepsilon_{m-1,p}$ and the power of 3^m in the denominator is increased, we obtain (4.10) with the factor $1/5$. The rest of the argument is the same as in Lemma 4.3. □

Conjecture 4.5. *For* $\alpha_2 + d/p < \alpha < \alpha_2 + \alpha_1$, $f \in (\Lambda_\alpha^{p,q})^{(3)}(SG)$ *if and only if* $f \in dom\,\Delta$ *and* $\Delta f \in (\Lambda_{\alpha-\alpha_2}^{p,q})^{(1)}(SG)$.

Note that by Lemmas 4.2(b) and 2.4(b), $(\Lambda_\alpha^{p,q})^{(3)}(SG) \subseteq dom\,\Delta$ if $\alpha > \alpha_2 + d/p$. There are technical problems with extending the proof of either half of Lemma 2.4(b) to prove the conjecture.

Definition 4.6. For $\alpha > d/p$, define k to be the nonnegative integer satisfying $k\alpha_2 + d/p < \alpha \leqslant (k+1)\alpha_2 + d/p$. Define $\Lambda_\alpha^{p,q}(SG)$ to be the space of bounded continuous functions u such that (if $k > 0$) $u \in dom\,(\Delta^k)$ and $\Delta^k u \in (\Lambda_{\alpha-k\alpha_2}^{p,q})^{(3)}(SG)$.

5. Heat equation characterizations

We expect that all function spaces described in this paper will be characterized in terms of solutions of the heat equation with given initial conditions. In this section we present the limited results we have been able to obtain so far. Given a function f on SG belonging to some L^p class, we may form the solution to the heat equation with Neumann boundary conditions $u(x,t) = e^{-t\Delta}f(x)$ given by (3.7).

Definition 5.1. Let k denote a positive integer, and assume $0 < \alpha < k\alpha_2$. Define $\tilde{\Lambda}_\alpha^{(k)}(SG)$ to be the space of bounded continuous functions f such that u satisfies

$$\left| \left(\frac{\partial}{\partial t}\right)^k u(x,t) \right| \leqslant M t^{-k+\frac{\alpha}{\alpha_2}} \quad \text{for } 0 < t \leqslant 1, \tag{5.1}$$

with the obvious norm.

Lemma 5.2. *Suppose* $\alpha < k\alpha_2$ and $j > k$. *Then* $f \in \tilde{\Lambda}_\alpha^{(j)}(SG)$ *if and only if* $f \in \tilde{\Lambda}_\alpha^{(k)}(SG)$.

Proof. Without loss of generality $j = k + 1$. First assume $f \in \tilde{\Lambda}_\alpha^{(j)}(SG)$. Use

$$\left(\frac{\partial}{\partial t}\right)^k u(x,t) = -\int_t^1 \left(\frac{\partial}{\partial s}\right)^{k+1} u(x,s)\, ds + \left(\frac{\partial}{\partial t}\right)^k u(x,1)$$

and the estimate (5.1) for $k+1$ to obtain

$$\left|\left(\frac{\partial}{\partial t}\right)^k u(x,t)\right| \leqslant M \int_t^1 s^{-k-1+\frac{\alpha}{\alpha_2}}\, ds + \left|\left(\frac{\partial}{\partial t}\right)^k u(x,1)\right|.$$

From the boundedness of f and estimate (3.8) for $t^k(\frac{\partial}{\partial t})^k p_t(x,y)$ we obtain a constant bound for $(\frac{\partial}{\partial t})^k u(x,1)$, and (5.1) follows since we consider only $t \leqslant 1$.

Conversely, assume $f \in \tilde{\Lambda}_\alpha^{(k)}(SG)$. It suffices to prove (5.1) with k replaced by j and t replaced by $2t$. By the semigroup property

$$\begin{aligned}\left|\left(\frac{\partial}{\partial t}\right)^j u(x,2t)\right| &= \left|\int \frac{\partial}{\partial t} p_t(x,y) \left(\frac{\partial}{\partial t}\right)^k u(y,t)\, d\mu(y)\right| \\ &\leqslant Mt^{-j+\frac{\alpha}{\alpha_2}} \int \left| t\frac{\partial}{\partial t} p_t(x,y)\right| d\mu(y).\end{aligned}$$

But this last integral is uniformly bounded because we have estimate (3.8) for the integrand. $\square$

Lemma 5.3. *For $\alpha < 1$ we have $\Lambda_\alpha^{(1)}(SG) \subseteq \tilde{\Lambda}_\alpha^{(1)}(SG)$.*

Proof. Let $f \in \Lambda_\alpha^{(1)}(SG)$. Then

$$\frac{\partial}{\partial t} u(x,t) = \int \frac{\partial}{\partial t} p_t(x,y)(f(y) - f(x))\, d\mu(y)$$

because $\int p_t(x,y)\, d\mu(y) = 1$. Now use (2.5) to estimate $(f(y) - f(x))$ and (3.8) to estimate $t\frac{\partial}{\partial t} p_t(x,y)$ to obtain

$$\left|\frac{\partial}{\partial t} u(x,t)\right| \leqslant ct^{-1-\frac{d}{\alpha_2}} \int d(x,y)^\alpha \exp\left(-c\left(\frac{d(x,y)^{\alpha_2}}{t}\right)^{1/d}\right) d\mu(y). \tag{5.2}$$

Now break up the integral in (5.2) into the region where $d(x,y) \leqslant t^{1/\alpha_2}$ and its complement. In the first region we pick up a factor of t^{α/α_2} from $d(x,y)^\alpha$ and t^{d/α_2} from the measure to obtain a bound of $ct^{-t+\alpha/\alpha_2}$ as desired. In the complementary region the exponential term swamps all the others, and a routine annular shell argument produces the same estimate. $\square$

Lemma 5.4. *For $\alpha < \alpha_2$ we have $\tilde{\Lambda}_\alpha^{(1)}(SG) \subseteq \Lambda_\alpha^{(2)}(SG)$.*

Proof. For $x \in V_m$ we write

$$\begin{aligned} \Delta_m f(x) &= (\Delta_m f(x) - \Delta_m u(x,t)) + \Delta_m u(x,t) \\ &= -\Delta_m \int_0^t \frac{\partial}{\partial s} u(x,s)\, ds + \Delta_m u(x,t). \end{aligned} \tag{5.3}$$

This is valid for all t, but we will make the specific choice $t = 5^{-m}$. For $f \in \tilde{\Lambda}_\alpha^{(1)}(SG)$ we have

$$\left| \Delta_m \int_0^t \frac{\partial}{\partial s} u(x,s) ds \right| \leqslant c \int_0^t s^{-1+\frac{\alpha}{\alpha_2}}\, ds \leqslant c t^{\alpha/\alpha_2} = c\left(\frac{3}{5}\right)^{m\alpha}.$$

To estimate the remaining term in (5.3) we use (2.17) and the fact that $u(x,t)$ satisfies the heat equation to obtain

$$\begin{aligned} \Delta_m u(x,t) &= c\left(\frac{3}{5}\right)^m \int \Delta u(y,t) \psi_x^{(m)}(y)\, d\mu(y) \\ &= c\left(\frac{3}{5}\right)^m \int \frac{\partial}{\partial t} u(y,t) \psi_x^{(m)}(y)\, d\mu(y). \end{aligned}$$

Now $\psi_x^{(m)}$ is bounded by one and the measure of its support is $2 \cdot 3^{-m}$. Thus

$$|\Delta_m u(x,t)| \leqslant c 5^{-m} t^{-1+\alpha/\alpha_2} = \left(\frac{3}{5}\right)^{m\alpha}. \qquad \square$$

Theorem 5.5. $\Lambda_\alpha(SG) = \tilde{\Lambda}_\alpha^{(1)}(SG)$ *for* $0 < \alpha < 1$.

Proof. If $f \in \tilde{\Lambda}_\alpha^{(1)}$ then $f \in \Lambda_\alpha^{(2)} = \Lambda_\alpha$ by Lemma 5.4. (This part of the argument only requires $\alpha < \alpha_2$.) Conversely, if $f \in \Lambda_\alpha = \Lambda_\alpha^{(1)}$ then $f \in \tilde{\Lambda}_\alpha^{(1)}$ by Lemma 5.3. $\square$

Lemma 5.6. *Suppose* $k\alpha_2 < \alpha < k\alpha_2 + 1$. *Then* $\Lambda_\alpha(SG) = \tilde{\Lambda}_\alpha^{(k+1)}(SG) + \mathscr{H}_{k-1}$.

Proof. Consider first the case $k = 1$. Since $\alpha > \alpha_2$, $\Lambda_\alpha(SG) \subseteq dom\, \Delta$, so any $f \in \Lambda_\alpha(SG)$ satisfies $\sum_{j=1}^3 \partial_n f(q_j) = 0$. Therefore, by subtracting a suitable harmonic function we may arrange to have $\partial_n f(q_j) = 0$ for $j = 1, 2, 3$. Under this assumption it follows that

$$\Delta f(x) = \sum_{j=1}^{\infty} -\lambda_j c_j \varphi_j(x)$$

when

$$f(x) = \sum_{j=1}^{\infty} c_j \varphi_j(x)$$

is the expansion of f in Neumann eigenfunctions. We then have

$$\begin{aligned}\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) &= \int p_t(x,y) \Delta f(y)\, d\mu(y) \\ &= \sum_{j=1}^{\infty} -\lambda_j e^{-\lambda_j t} c_j \varphi_j(x).\end{aligned} \tag{5.4}$$

Since $\Delta f \in \Lambda_{\alpha-\alpha_2}(SG)$, Theorem 5.5 implies $|(\frac{\partial}{\partial t})^2 u(x,t)| \leqslant M t^{-1+(\alpha-\alpha_2)/\alpha_2}$, which means $f \in \tilde{\Lambda}_\alpha^{(2)}(SG)$.

Since $\mathscr{H}_0 \subseteq \Lambda_\alpha(SG)$ for any α, for the converse we need to show $\tilde{\Lambda}_\alpha^{(2)}(SG) \subseteq \Lambda_\alpha(SG)$. First we claim $\tilde{\Lambda}_\alpha^{(2)}(SG) \subseteq dom\, \Delta$. For this we observe

$$\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial t} u(x,1) - \int_t^1 \left(\frac{\partial}{\partial s}\right)^2 u(x,s)\, ds. \tag{5.5}$$

The condition $f \in \tilde{\Lambda}_\alpha^{(2)}(SG)$ implies that the integral $\int_0^1 (\frac{\partial}{\partial s})^2 u(x,s)\, ds$ is absolutely convergent. By routine arguments we may pass to the limit $t \to 0$ in (5.5) to obtain

$$\Delta f(x) = \frac{\partial}{\partial t} u(x,1) - \int_0^1 \left(\frac{\partial}{\partial s}\right)^2 u(x,s)\, ds$$

so $f \in dom\, \Delta$. Again we may subtract a harmonic function to make the normal derivatives of f vanish at the boundary, and so (5.4) holds again. The condition $f \in \tilde{\Lambda}_\alpha^{(2)}(SG)$ implies $\Delta f \in \Lambda_{\alpha-\alpha_2}^{(1)}(SG)$, so Theorem 5.5 yields $\Delta f \in \Lambda_{\alpha-\alpha_2}(SG)$, hence $f \in \Lambda_\alpha(SG)$.

The result for general k is proved by iterating the same arguments. □

Conjecture 5.7. *For $\alpha < (k+1)\alpha_2$ we have $\Lambda_\alpha(SG) = \tilde{\Lambda}_\alpha^{(k+1)}(SG) + \mathscr{H}_k$.*

This conjecture is a bit sloppy, since we know for $\alpha < k\alpha_2 + 1$ we may reduce $\mathscr{H}_k$ to $\mathscr{H}_{k-1}$. It is not clear where the transition from $\mathscr{H}_{k-1}$ to $\mathscr{H}_k$ occurs. The way the conjecture is formulated, we know it is true when $j\alpha_2 < \alpha < j\alpha_2 + 1$ for $j \leqslant k$, so it is conceivable that interpolation methods might yield a proof. We also know the containment

$$\tilde{\Lambda}_\alpha^{(k+1)}(SG) + \mathscr{H}_k \subseteq \Lambda_\alpha(SG)$$

for $\alpha \neq j\alpha_2$ because of Lemma 5.4. To prove the reverse containment appears to be difficult, however, since we do not have any analog of Lemma 2.2 to use to trade information about the discrete $\Delta_m f(x)$ for information about $\Delta u(x,t)$.

As far as the Besov spaces are concerned, it is clear that the analog of condition (5.1) should be

$$\begin{cases} \int_0^1 \left\| \left(\frac{\partial}{\partial t}\right)^k u(\cdot,t) \right\|_p^q t^{q(k-\frac{\alpha}{\alpha_2})-1} dt < \infty & \text{for } 1 \leqslant q < \infty, \\ \sup_{0<t\leqslant 1} \left\| \left(\frac{\partial}{\partial t}\right)^k u(\cdot,t) \right\|_p t^{k-\frac{\alpha}{\alpha_2}} < \infty & \text{for } q = \infty. \end{cases} \tag{5.6}$$

Using (5.6) we would not have to make the restriction $\alpha > d/p$. However, we do not know how to prove that (5.6) characterizes $\Lambda_\alpha^{p,q}(SG)$ for any values of α.

6. Extensions and restrictions

Given a subset of SG and one of the function spaces already defined, we can consider the restrictions of functions from the space to the subset. For simplicity, we only deal with the case when the subset is a cell $F_w SG$. We could also handle finite unions of cells by similar methods, but more complicated subsets present a greater challenge. If we restrict a function u to $F_w SG$ and then blow it up to SG by composing with F_w, the result is simply $u \circ F_w$. We will call this the *restriction operator*, by a slight abuse of notation. Similarly, by an *extension operator* we mean an operator E satisfying $(Ev)\circ F_w = v$.

Theorem 6.1. *The restriction operator is bounded on all the function spaces previously defined.*

Proof. For the Hölder–Zygmund and Besov spaces this is an immediate consequence of the definition. For the Sobolev spaces we prove the result first for the case $s = k\alpha_2$, where by Theorem 3.9 we may identify $L^p_{k\alpha_2}$ with the functions u such that $\Delta^k u \in L^p$. It is clear that this is preserved under restriction. The general result follows by interpolation. $\square$

The interesting question is whether the restriction operator is onto, and if there exist linear extension operators. One possible extension operator is the even flip EF defined next. For simplicity we take $w = (1)$, but it is clear that we can iterate this procedure.

Definition 6.2. If u is defined on $F_1 SG$, the even flip EFu is defined by simultaneously taking the even extension across the points $F_1 q_2$ and $F_1 q_3$ and the reflections on $F_2 SG$ and $F_3 SG$ along the symmetry axis passing through these points, as illustrated in Fig. 3. Note that if u is continuous on $F_1 SG$ then EFu is continuous on SG.

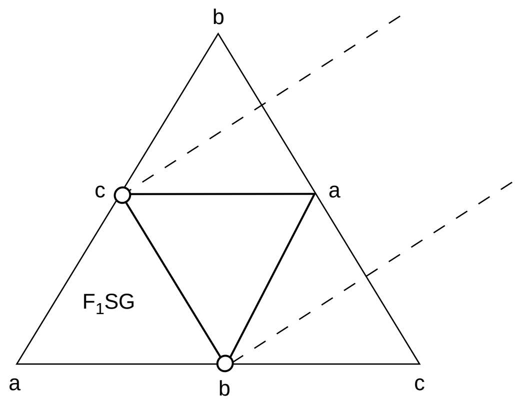

Fig. 3. The even flip extends the function defined on F_1SG with indicated boundary values a, b, c by even extension about the circled points and reflection in the symmetry axes as shown.

Lemma 6.3. *The mapping $v \to EF(v \circ F_1^{-1})$ is bounded on $L^p_{s,N}(SG)$ for $0 \leqslant s \leqslant \alpha_2$.*

Proof. By interpolation (Corollary 3.4) it suffices to show this for $s = 0$ and α_2, and for $s = 0$ it is trivial since $L^p_{0,N}(SG) = L^p$. For $s = \alpha_2$ we use the characterization of $L^p_{\alpha_2,N}(SG)$ as all v satisfying $\Delta v \in L^p$ and having vanishing normal derivatives (Lemma 3.5). The same is true for $v \circ F_1^{-1}$, and so we have both matching values and normal derivatives at junction points to obtain $\Delta EF(v \circ F_1^{-1}) = 5EF(\Delta v \circ F_1^{-1})$, and the result follows. $\square$

Theorem 6.4. *For every $s \geqslant 0$ the restriction map $u \to u \circ F_w$ maps $L^p_s(SG)$ onto itself, and there exists a bounded linear extension map E_s on L^p_s.*

Proof. Again we may assume $w = (1)$. First consider the case $0 \leqslant s \leqslant \alpha_2$. If $\mathscr{H}_0 \subseteq L^p_{s,N}(SG)$ then $L^p_s(SG) = L^p_{s,N}(SG)$ and we may take $E_s v = EF(v \circ F_1^{-1})$. If not, choose a complementary subspace $\tilde{\mathscr{H}}_0$ to $\mathscr{H}_0 \cap L^p_{s,N}(SG)$ in $\mathscr{H}_0$, so $L^p_s(SG) = L^p_{s,N}(SG) + \tilde{\mathscr{H}}_0$ (disjoint). Since $\tilde{\mathscr{H}}_0$ is a finite-dimensional space, there exists a bounded projection P of $L^p_s(SG)$ onto $\tilde{\mathscr{H}}_0$, so that $(I - P)$ maps $L^p_s(SG)$ onto $L^p_{s,N}(SG)$. We may then take

$$E_s v = EF(((I - P)v) \circ F_1^{-1}) + EH(Pv \circ F_1^{-1}),$$

where EH is the harmonic extension mapping from F_1SG to SG [DSV].

In general, $k\alpha_2 \leqslant s \leqslant (k+1)\alpha_2$ for some positive integer k. If $v \in L^p_{s,N}(SG)$, then $\Delta^k v \in L^p_{s-k\alpha_2,N}(SG)$, and so $EF((\Delta^k v) \circ F_1^{-1})$ is bounded from $L^p_{s,N}(SG)$ to $L^p_{s-k\alpha_2,N}(SG)$, and $w = (-1)^k(-\Delta_D)^{-k} EF((\Delta^k v) \circ F_1^{-1})$ is bounded from $L^p_{s,N}(SG)$ to

$L^p_s(SG)$. This is not quite an extension operator because we only know that $\Delta^k(w\circ F_1) = \Delta^k v$, so $w\circ F_1$ differs from v by a function in $\mathscr{H}_{k-1}$. It is clear that a linear operator $L_{k-1} : L^p_{s,N}(S) \to \mathscr{H}_{k-1}$ gives the difference, so $v \to w - L_{k-1}v$ is an extension operator, bounded from $L^p_{s,N}(SG)$ to $L^p_s(SG)$.

Now we can complete the proof as in the first case, since $L^p_s(SG) = L^p_{s,N}(SG) + \tilde{\mathscr{H}}_k$ (disjoint) for some subspace of $\mathscr{H}_k$, which is finite dimensional, and there is a linear extension map on $\mathscr{H}_k$ (this is proved in [SU]). $\square$

A somewhat technical issue is whether it is possible to find a single extension operator that will work for all s as in [SE] or even for all s in a bounded interval. Another interesting question is to characterize the subspace of $L^p_s(SG)$ for which the zero extension operator maps into $L^p_s(SG)$. It is easy to see that for $s = \alpha_2$ the answer is: all functions vanishing together with their normal derivatives at the two relevant boundary points. In analogy with the interval, we would expect the subspace to have finite codimension and be characterized by a finite number of vanishing conditions at the relevant boundary points for s in certain open intervals, depending on p, and by certain integral conditions for the special s values on the boundary of those intervals [S1].

Turning to the Hölder–Zygmund and Besov spaces, it is clear that the even flip is a bounded extension operator on $\Lambda^{(1)}_\alpha(SG)$ and $(\Lambda^{p,q}_\alpha)^{(1)}(SG)$, so this suffices to handle $\Lambda_\alpha(SG)$ for $\alpha < 1$ and $\Lambda^{p,q}_\alpha(SG)$ if (4.5) holds. For the Hölder–Zygmund spaces we can go up to $\alpha < 2$ using an odd extension and Theorem 2.8 to localize.

Theorem 6.5. *There exists an extension operator bounded on all $\Lambda_\alpha(SG)$ for $\alpha < 2$.*

Proof. By subtracting off a harmonic function we may arrange for v to vanish at the relevant boundary points. Let w denote the odd extension of $v\circ F_1^{-1}$ across each of F_1q_2 and F_1q_3 into F_2SG and F_3SG. It is straightforward to see that the $\Lambda^{(2)}_\alpha(SG)$ estimate for v yields the same estimate at all interior points of F_2SG and F_3SG. At the reflection points F_1q_2 and F_1q_3 we have that $\Delta_m w$ vanishes because the extension is odd, so the estimate is trivial. However, at the junction point $F_2q_3 = F_3q_2$ the $\Lambda^{(2)}_\alpha$ estimate fails. Thus $v \to w$ is not quite the desired extension.

We correct the problem by multiplying by a cutoff function ψ. We need ψ to be identically one on F_1SG so that $v \to \psi w$ is still an extension operator, and ψ must vanish in a neighborhood of the problematic junction point. In [SU] it is explained how to do this using splines, and if we take the order sufficiently large (triharmonic will certainly suffice) then ψ will be in *dom* Δ, hence in all $\Lambda_\alpha(SG)$ for $\alpha < 2$. Then ψw is in $\Lambda_\alpha(SG)$ by Theorem 2.8. $\square$

By using the same ideas as in the proof of Theorem 6.4 we may extend the result to all α satisfying $\alpha = k\alpha_2 + s$ for $0 < s < 2$. Unfortunately, this leaves gaps of the form $k\alpha_2 + s$ for $2 \leqslant s \leqslant \alpha_2$. It seems likely that the best way to define extension operators is to glue on splines based on boundary values and derivatives of the function.

However, there are some technical problems that need to be resolved to implement this approach.

7. Other fractals

We may carry over much of the theory from SG to a fairly general pcf fractal with a regular self-similar energy form. To avoid technical difficulties we make slightly stronger assumptions than in [Ki]. We assume K is a connected self-similar invariant set for an ifs of contractive similarities $\{F_i\}$ on some Euclidean space $\mathbb{R}^n$, so

$$K = \bigcup_{i=1}^{N} F_i K. \tag{7.1}$$

The boundary V_0 of K consists of the fixed points q_i of the first N_0 mappings F_i. The key assumption is

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0 \quad \text{for } i \neq j, \tag{7.2}$$

so the cells $F_i K$ intersect at images of boundary points only. We will call such a set K a pcf fractal.

To describe the regular self-similar energy form on K we assume that we are given a vector $(r_1, \ldots, r_N)$ of numbers in $(0,1)$ that may be interpreted as resistance contraction factors for the mappings $\{F_i\}$. They will be approximately the contraction factors associated with the mappings for the resistance metric. However, they need not have any connection with the similarity contraction factors in the Euclidean metric. Let $w = (w_1, \ldots, w_m)$ denote a word with each $w_k \in \{1, \ldots, N\}$. We will use a standard "stopping time" approach to create collections $\mathscr{W}_n$ of words with r_w on the order of $(r_{\max})^n$, where $r_{\max}$ and $r_{\min}$ denote the maximum and minimum values of $\{r_i\}$, and $r_w = r_{w_1} r_{w_2} \cdots r_{w_m}$. More precisely, we will have

$$r_{\min}(r_{\max})^n \leqslant r_w \leqslant (r_{\max})^n \tag{7.3}$$

for every word in $\mathscr{W}_n$. However, not every word satisfying (7.3) is in $\mathscr{W}_n$, and a particular word may belong to $\mathscr{W}_n$ for several different values of n. We take $\mathscr{W}_1$ to be all singleton words (then (7.3) is obvious). Inductively, having defined $\mathscr{W}_n$ satisfying (7.3), we define $\mathscr{W}_{n+1}$ by retaining all words from $\mathscr{W}_n$ satisfying $r_w \leqslant (r_{\max})^{n+1}$, and replacing each w in $\mathscr{W}_n$ for which the reverse inequality holds by all N words obtained by adding one letter to w. It is easy to see that (7.3) with n replaced by $n+1$ holds for all words in $\mathscr{W}_{n+1}$.

We iterate (7.1) to obtain

$$K = \bigcup_{w \in \mathscr{W}_n} F_w K, \tag{7.4}$$

which we will call the decomposition of K into cells of level n, and we let

$$V_n = \bigcup_{w \in \mathscr{W}_n} F_w V_0, \tag{7.5}$$

which we will call the vertices of level n. The graph Γ_n of level n has vertices V_n and edge relation $x \sim_n y$ if x and y belong to the same cell of level n. Note that when we pass from level n to level $n+1$, some cells $F_w K$ will remain unchanged, and some will subdivide into N cells $F_w F_i K$.

Now we assume we are given an energy form

$$\mathscr{E}_0(f, f) = \sum_{j<k} c_{jk}(f(q_j) - f(q_k))^2 \tag{7.6}$$

on Γ_0, where the conductance coefficients c_{jk} are nonnegative, and enough of them are positive so that $\mathscr{E}_0(f, f) = 0$ only if f is constant on V_0. We define the energy form

$$\mathscr{E}_n(f, f) = \sum_{w \in \mathscr{W}_n} r_w^{-1} \mathscr{E}_0(f \circ F_w, f \circ F_w) \tag{7.7}$$

on Γ_n. If f is defined on V_0, we call $\tilde{f}$ the *harmonic extension* to V_n if it minimizes $\mathscr{E}_n$. The key assumption is that $\mathscr{E}_n(\tilde{f}, \tilde{f}) = \mathscr{E}_0(f, f)$. It is easy to see that it suffices to check this for $n = 1$. We call $(\mathscr{E}_0, \{r_i\})$ a *regular harmonic structure*. In that case we may define

$$\mathscr{E}(f, f) = \lim_{n \to \infty} \mathscr{E}_n(f, f) \in [0, \infty] \tag{7.8}$$

for any continuous function on K, and we define *dom* $\mathscr{E}$ to be all functions of finite energy, $\mathscr{E}(f, f) < \infty$. It follows easily that $\mathscr{E}(f, f)$ satisfies the self-similar identity

$$\mathscr{E}(f, f) = \sum_{w \in \mathscr{W}_n} r_w^{-1} \mathscr{E}(f \circ F_w, f \circ F_w). \tag{7.9}$$

Of course, the existence of regular harmonic structures is not an easy problem, but by now there are many interesting examples known.

The *resistance metric* $d(x, y)$ is defined by

$$d(x, y) = (\min\{\mathscr{E}(f, f) : f(x) = 0, f(y) = 1\})^{-1}. \tag{7.10}$$

By considering piecewise harmonic splines of level n it is easy to see that for $x \sim_n y$ neighboring vertices of level n, $d(x, y)$ is bounded above and below by multiples of $(r_{\max})^n$, and in fact a cell $F_w K$ of order n has diameter comparable to $(r_{\max})^n$. It is shown in [Ki] that the resistance metric is indeed a metric, and it induces the same topology on K as the Euclidean metric. Note that we may not have a metric equivalence relating $d(x, y)$ with a power of the Euclidean distance, as was the case for SG (see [Mo]).

In order to define a Laplacian we need to choose a measure μ on K, and we will take μ to be a self-similar probability measure with weights $\{\mu_i\}$ related to $\{r_i\}$ by

$$\mu_i = r_i^d, \tag{7.11}$$

where the dimension d is defined by

$$\sum_{i=1}^{N} r_i^d = 1. \tag{7.12}$$

The measure μ is determined by the identity

$$\mu = \sum_{i=1}^{N} \mu_i \mu \circ F_i^{-1}, \tag{7.13}$$

which becomes

$$\mu = \sum_{w \in \mathscr{W}_n} \mu_w \mu \circ F_w^{-1} \tag{7.14}$$

on iteration. Note that a ball of radius R in the resistance metric will have measure comparable to R^d.

The weak formulation of the Laplacian is that $u \in dom\, \Delta$ with $\Delta u = f$ if u and f are continuous, $u \in dom\, \mathscr{E}$ and

$$-\mathscr{E}(u, v) = \int fv\, d\mu \quad \text{for all } v \in dom_0\, \mathscr{E}, \tag{7.15}$$

where $\mathscr{E}(u, v)$ denotes the bilinear form associated to the quadratic form $\mathscr{E}(u, u)$, and the subscript 0 indicates the functions vanishing on V_0. There is also a pointwise formulation involving a discrete Laplacian Δ_n on Γ_n. Let

$$\Delta_0 f(q_j) = \sum_k c_{jk}(f(q_k) - f(q_j)), \tag{7.16}$$

and

$$\Delta_n f(x) = \sum_{\substack{x = F_w q_j \\ w \in \mathscr{W}_n}} (r_{\max})^n r_w^{-1} \Delta_0 (f \circ F_w)(q_j). \tag{7.17}$$

Note that the factor $(r_{\max})^n$ in (7.17) is a bit artificial, and is introduced to make the notation consistent with the previous notation for the case $K = SG$. Let $\psi_y^{(n)}$ for $y \in V_n$ be defined to be the piecewise harmonic spline of level n that satisfies

$$\psi_y^{(n)}(x) = \delta(x, y) \quad \text{for } x \in V_n. \tag{7.18}$$

Then

$$\Delta f(x) = \lim_{n\to\infty} \left((r_{\max})^{-n} \int \psi^{(n)}\, d\mu \right)^{-1} \Delta_n f(x) \tag{7.19}$$

for every $x \in V_* \setminus V_0$. More precisely, Kigami [Ki] proves that $f \in dom\, \Delta$ if and only if limit (7.19) exists uniformly. For such functions there exist normal derivatives

$$\partial_n f(q_j) = -\lim_{n\to\infty} (r_{\max})^n \Delta_n f(q_j) \tag{7.20}$$

at boundary points. Just as for SG, this can be localized (Definition 2.6), and the same results about matching conditions hold, as well as the Gauss–Green formula

$$\int_K (u\Delta v - v\Delta u)\, d\mu = \sum_{\partial K} (u\partial_n v - v\partial_n u). \tag{7.21}$$

In particular, this implies

$$\Delta_n u(y) = (r_{\max})^n \int \psi_y^{(n)} \Delta u\, d\mu. \tag{7.22}$$

We also note that $\int \psi_y^{(n)}\, d\mu$ is bounded above and below by multiples of $(r_{\max})^{nd}$.

We can now make the definitions analogous to Definition 2.1. Unfortunately, we do not know the analog of α_1, so we omit the upper bounds on α in (a) and (c). Therefore, for large enough α, these spaces become trivial.

Definition 7.1. (a) Define $\Lambda_\alpha^{(1)}(K)$ to be the Banach space of all bounded continuous functions on K satisfying

$$|f(x) - f(y)| \leqslant M(r_{\max})^{\alpha n} \quad \text{for } x, y \in V_n, x \sim_n y. \tag{7.23}$$

(b) For $0 < \alpha \leqslant \alpha_2 = d + 1$ define $\Lambda_\alpha^{(2)}(K)$ as before with (7.23) replaced by

$$|\Delta_n f(x)| \leqslant M(r_{\max})^{\alpha n} \quad \text{for } x \in V_n \setminus V_0. \tag{7.24}$$

(c) Define $\Lambda_\alpha^{(3)}(K)$ as before with

$$|\Delta_n f(x) - \Delta_n f(y)| \leqslant M(r_{\max})^{\alpha n} \quad \text{for } x, y \in V_n \setminus V_0, x \sim_n y. \tag{7.25}$$

It is straightforward to extend the proof of Lemma 2.2 to show that $\Lambda_\alpha^{(1)}(K)$ may be identified with the Hölder space of functions satisfying

$$|f(x) - f(y)| \leqslant Md(x, y)^\alpha \text{ for all } x, y \in K. \tag{7.26}$$

In order to obtain the analog of Lemma 2.3 we need to find analogs of the combinatorial identities (2.8) and (2.9).

Lemma 7.2. *Let F_wK denote one of the cells of level $n-1$ that subdivides $F_wK = \bigcup_{j=1}^N F_wF_jK$ in level n. Then for any x, y in $V_n \cap F_wK$ with $x \sim_n y$, there exist coefficients $c(z)$ and $b(x', y')$ such that*

$$f(x) - f(y) = \sum_{z \in (V_n \setminus V_{n-1}) \cap F_wK} c(z) \Delta f_n(z) + \sum_{x', y' \in V_{n-1} \cap F_wK} b(x', y')((f(x') - f(y')) \tag{7.27}$$

for all functions f defined on $V_n \cap F_wK$.

Proof. If $x \sim_n y$ then there exists j such that $x, y \in V_n \cap F_wF_jK$. It is easy to see that we cannot have both x and y in V_{n-1}. It is shown in Kigami [Ki] that the linear functionals $\Delta_n f(z)$ form a basis for the dual space of functions on $(V_n \setminus V_{n-1}) \cap F_wK$, so there exists a unique choice of coefficients $c(z)$ such that

$$f(x) - f(y) = \sum_{z \in (V_n \setminus V_{n-1}) \cap F_wK} c(z) \Delta_n f(z) + \sum_{x' \in V_{n-1} \cap F_wK} B(x') f(x'). \tag{7.28}$$

But if we choose $f \equiv 1$ then the left-hand side and the first term on the right-hand side of (7.28) vanish, hence $\sum B(x') = 0$. Thus we may rearrange the last terms in (7.28) to obtain the form (7.27). Note that the coefficients $b(x', y')$ are not unique. $\square$

In order to turn (7.27) into an estimate we need to understand what happens to harmonic functions, when the first term on the right-hand side of (7.27) or (7.28) vanishes. Recall that there is a harmonic extension algorithm that may be written

$$h(F_wF_iq_j) = \sum_{k=1}^N (A_i)_{jk} h(F_wq_k) \tag{7.29}$$

for any harmonic function h, and certain nonnegative matrices A_i. Note that this algorithm is independent of the cell F_wK, and (7.29) determines the coefficients $B(x')$ in (7.28). Now the matrix A_i reproduces the constant vector, so its largest eigenvalue is 1. In Appendix A1 of [Ki] it is shown that the second largest eigenvalue of A_i is $\leqslant r_i$. We will assume a slightly stronger condition. It is possible that this always holds, but in any case it holds for many examples.

Hypothesis 7.3. Assume that, for each i,

$$\sup_{j \neq k} |(A_i x)_j - (A_i x)_k| \leqslant r_i \sup_{j \neq k} |x_j - x_k| \tag{7.30}$$

for every vector x.

Lemma 7.4. *Under Hypothesis* 7.3, *there exists a constant c such that*

$$\begin{aligned} |f(x) - f(y)| \leqslant\; & c \sup\{|\Delta_n f(z)| : z \in (V_n \backslash V_{n-1}) \cap F_w K\} \\ & + r_i \sup\{|f(x') - f(y')| : x', y' \in V_{n-1} \cap F_w K\} \end{aligned} \tag{7.31}$$

for $x, y \in V_n \cap F_w F_i K$.

Proof. Write $f = h + g$ when h is harmonic and g vanishes on $V_{n-1} \cap F_w K$. Recall that $\Delta_n h(z) = 0$ so $\Delta_n f(z) = \Delta_n g(z)$. Then $|g(x) - g(y)|$ is bounded by the first term on the right-hand side of (7.31) by (7.27), with $c = \sum |c(z)|$. But $|h(x) - h(y)|$ is bounded by the second term on the right-hand side of (7.31) by (7.29) and (7.30). $\square$

It is then easy to check that the proof of Lemma 2.3 goes through with minor modifications, so

$$\Lambda_\alpha^{(1)}(K) = \Lambda_\alpha^{(2)}(K) \quad \text{for } 0 < \alpha < 1. \tag{7.32}$$

Next we consider the analog of identity (2.13). So let $x \in V_{n-1} \backslash V_0$ and let $U_{n-1}(x)$ be its neighborhood of level $n-1$, namely the union of all level $n-1$ cells that contain x.

Lemma 7.5. *There exists a constant* $b_n(x) \leqslant r_{\max}$ *and constants* $c(y, z)$ *such that*

$$\Delta_n f(x) - b_n(x) \Delta_{n-1} f(x) = \sum_{\substack{y \sim_n z \\ y, z \in U_{n-1}(x) \cap V_n}} c(y, z)(\Delta_n f(y) - \Delta_n f(z)). \tag{7.33}$$

Proof. We use (7.22) to obtain

$$\Delta_n f(x) - b \Delta_{n-1} f(x) = (r_{\max})^n \int \left(\psi_x^{(n)} - \frac{b}{r_{\max}} \psi_x^{(n-1)} \right) \Delta f \, d\mu. \tag{7.34}$$

Note that $\psi_x^{(n)} - \frac{b}{r_{\max}} \psi_x^{(n-1)}$ is a piecewise harmonic spline of level n with support in $U_{n-1}(x)$, so it is a linear combination of the functions $\psi_y^{(n)}$ for $y \in U_{n-1}(x) \cap V_n$. We will have

$$\psi_x^{(n)} - \frac{b}{r_{\max}} \psi_x^{(n-1)} = \sum c(y,z)(\psi_y^{(n)} - \psi_z^{(n)}) \tag{7.35}$$

if and only if

$$\sum_{y \in U_{n-1}(x) \cap V_n} \left(\psi_x^{(n)}(y) - \frac{b}{r_{\max}} \psi_x^{(n-1)}(y) \right) = 0. \tag{7.36}$$

This leads to the choice

$$b = b_n(x) = r_{\max} \left(\sum_{y \in U_{n-1}(x) \cap V_n} \psi_x^{(n-1)}(y) \right)^{-1} \tag{7.37}$$

which makes (7.36) hence (7.35) hold. Substituting (7.35) in (7.34) and again using (7.22) we obtain (7.33). From (7.37) we see that $b_n(x) \leqslant r_{\max}$. $\square$

In order to simplify (7.37) we make a rather strong assumption.

Hypothesis 7.6. All r_i are equal and $\int \psi_x^{(n)}\, d\mu$ is independent of $x \in V_n \backslash V_0$.

Lemma 7.7. *Under Hypothesis* 7.6,

$$b_n(x) = (r_{\max})^{\alpha_2} \quad \textit{for } x \in V_{n-1} \backslash V_0. \tag{7.38}$$

Proof. Since $\psi_x^{(n-1)}$ is also a piecewise harmonic spline of level n, we have

$$\psi_x^{(n-1)} = \sum_{y \in V_n} \psi_x^{(n-1)}(y) \psi_y^{(n)}. \tag{7.39}$$

We integrate (7.39) and use Hypothesis 7.6 to obtain

$$\int \psi_x^{(n-1)}\, d\mu = \left(\sum_{y \in V_n} \psi_x^{(n-1)}(y) \right) \int \psi_x^{(n)}\, d\mu.$$

The hypothesis also implies

$$\int \psi_x^{(n)}\, d\mu = (r_{\max})^d \int \psi_x^{(n-1)}\, d\mu,$$

so

$$\left(\sum_{y\in V_n}\psi_x^{(n-1)}(y)\right)^{-1}=(r_{\max})^d,$$

and (7.38) follows since $\alpha_2=d+1$. $\square$

Lemma 7.8. *Assume Hypothesis* 7.6(a) $\Lambda_\alpha^{(3)}(K)=\Lambda_\alpha^{(2)}(K)$ *for* $0<\alpha<\alpha_2$. (b) *For* $\alpha_2<\alpha$, $\psi\in\Lambda_\alpha^{(3)}(K)$ *if and only if* $u\in dom\,\Delta$ *and* $\Delta u\in\Lambda_{\alpha-\alpha_2}^{(1)}(K)$.

Proof. (a) For $x\in V_{n-1}\backslash V_0$ we use (7.33) in place of (2.13), while for $x\in V_n\backslash V_{n-1}$ we use the same trick as in (2.14), to obtain

$$\varepsilon_n\leqslant(r_{\max})^{\alpha_2}\varepsilon_{n-1}+c\eta_n \tag{7.40}$$

in place of (2.16). The rest of the proof of Lemma 2.4(a) is the same.

(b) Assume $u\in dom\,\Delta$ and $\Delta u\in\Lambda_{\alpha-\alpha_2}^{(1)}(K)$. Use (7.22) in place of (2.17) to obtain, for $x\sim_n y$,

$$\begin{aligned}|\Delta_n u(x)-\Delta_n u(y)|&=(r_{\max})^n\left|\int\psi_x^{(n)}\Delta u\,d\mu-\int\psi_y^{(n)}\Delta u\,d\mu\right|\\&=(r_{\max})^n\left(\int\psi_x^{(n)}\,d\mu\right)|\Delta u(x')-\Delta u(y')|\end{aligned}$$

for some $x'\in U_n(x)$ and $y'\in U_n(y)$. But $d(x',y')\leqslant c(r_{\max})^n$ so $|\Delta u(x')-\Delta u(y')|\leqslant c(r_{\max})^{n(\alpha-\alpha_2)}$ and $\int\psi_x^{(n)}\,d\mu\leqslant c(r_{\max})^{nd}$, so $u\in\Lambda_\alpha^{(3)}(K)$ since $\alpha_2=d+1$.

Conversely, assume $u\in\Lambda_\alpha^{(3)}(K)$ and fix $x\in V_n\backslash V_0$. For every $k\geqslant1$ we can use (7.33) to estimate

$$\begin{aligned}&\left|(r_{\max})^{-(n+k)}\left(\int\mu_x^{(n+k)}\,d\mu\right)^{-1}\Delta_{n+k}u(x)\right.\\&\qquad\left.-(r_{\max})^{-(n+k-1)}\left(\int\psi_x^{(n+k-1)}\,d\mu\right)^{-1}\Delta_{n+k-1}u(x)\right|\\&\qquad\leqslant c(r_{\max})^{-(n+k)d}\eta_{n+k}.\end{aligned}\tag{7.41}$$

We may use (7.41) in place of (2.18), and the rest of the proof is the same as in Lemma 2.4(b). $\square$

Under Hypotheses 7.3 and 7.6, we may define $\Lambda_\alpha(K)$ as in Definition 2.5. The remainder of the results in Section 2 remain valid. For Theorem 2.7, we need the analog of (2.22). Since

$$(r_{\max})^{-n}\Lambda_n f(q_j) = \int \psi_{q_j}^{(n)} \Delta u \, d\mu$$

we have

$$(r_{\max})^{-n}\Lambda_n f(q_j) - (r_{\max})^{-n+1}\Lambda_{n-1} f(q_j) = \int (\psi_{q_j}^{(n)} - \psi_{q_j}^{(n-1)})\Delta u \, d\mu.$$

But we can write

$$\psi_{q_j}^{(n)} - \psi_{q_j}^{(n-1)} = \sum_{V_n \setminus V_0} c(y)\psi_y^{(n)}$$

with a uniform bound on $\sum_{V_n \setminus V_0} |c(y)|$. Thus

$$\begin{aligned}(r_{\max})^{-n}\Lambda_n f(q_j) - (r_{\max})^{-n+1}\Lambda_{n-1} f(q_j)\\ = (r_{\max})^{-n}\sum_{V_n \setminus V_0} c(y)\Lambda_n f(y),\end{aligned} \tag{7.42}$$

and this will serve in place of (2.22).

It is interesting to observe what happens when K is the unit interval, which may be regarded as a pcf self-similar fractal with $V_n = \{k2^{-n} : 0 \leqslant k \leqslant 2^n\}$. Then the $\Lambda_\alpha(K)$ spaces are just the dyadic versions of the usual Hölder–Zygmund spaces. It is shown by Ciesielski [C] that these coincide with the usual Hölder–Zygmund spaces for all $\alpha > 0$, and this result has been extended by Kamont [Ka] to include Besov spaces for $\alpha > 1/p$. The equivalence for Hölder–Zygmund spaces when α is not an integer is relatively easy to see, but is nontrivial when α is an integer. For example, when $\alpha = 1$, the result says that

$$|f(x+2h) - 2f(x+h) + f(x)| \leqslant M|h|$$

when $h = 2^{-n}$ and $x = k2^{-n}$, for all n and $0 \leqslant k \leqslant 2^n - 2$ implies the same condition for all x and h. An interesting observation in this regard is that the dyadic second differences of f may be identified with the Haar basis coefficients of f' (with a factor of $2^{n/2}$). The dyadic Zygmund space $\Lambda_1^{(2)}(K)$ thus consists of the functions f such that f' has bounded Haar coefficients. One needs to be a bit careful here to make this precise, since f' only exists as a distribution, and Haar coefficients are not defined for all distributions (the mild singularity of f' puts it into a Sobolev class $L^2_{-\varepsilon}$ for small ε, and the Haar functions belong to L^2_ε for $\varepsilon < 1/2$, so that allows you to define the Haar coefficients). So the above discussion means that distributions in the mysterious Λ_0 space are characterized by having bounded Haar coefficients.

We consider next the results in Section 3. One of the key tools was the estimate (3.8) for the heat kernel. In [HK] it is shown that a similar estimate holds in general, except that the exponent $1/d$ inside the exponential must be replaced by a more complicated expression that involves the "chemical exponent" that may depend on x and y. However, this modification does not appear to make a difference for the applications. Also, although it is not explicitly stated in [HK], the same estimates hold for $t^k(\frac{\partial}{\partial t})^k p_t(x,y)$. All the results of Section 3 remain valid with essentially the same proofs. The only one that requires comment is Lemma 3.12(b). Here the formula for the Green's function (3.18) becomes

$$G(x,y) = \sum_{n=0}^{\infty} \sum_{w \in \mathscr{W}_n} r_w \Psi_w(x,y). \tag{7.43}$$

We then obtain $\Delta_n G(x,y) = 0$ for $y \notin U_n(x)$ and $x \in V_n$, and $|\Delta_n G(x,y)| \leqslant c(r_{\max})^n$. Since

$$\Delta_n u(x) = \int_{U_n(x)} \Delta_n G(x,y) f(y)\, d\mu(y),$$

we obtain the estimate

$$|\Delta_n u(x)| \leqslant c(r_{\max})^n \int_{U_n(x)} |f(y)|\, d\mu(y)$$

and Hölder's inequality yields

$$|\Delta_n u(x)| \leqslant c(r_{\max})^n \mu(U_n(x))^{1/p'} ||f||_p. \tag{7.44}$$

Also $\mu(U_n(x)) \sim (r_{\max})^{nd}$, so (7.44) implies $u \in \Lambda^{(2)}_{\alpha_2 - d/p}(K)$.

For the results of Section 4, in Definition 4.1 we replace (4.1) by

$$\begin{cases} \left(\sum_{n=0}^{\infty} (\delta_{n,p}(r_{\max})^{-n\alpha})^q\right)^{1/q} \leqslant M & \text{if } q < \infty, \\ \sup_n \delta_{n,p}(r_{\max})^{-n\alpha} \leqslant M & \text{if } q = \infty, \end{cases} \tag{7.45}$$

we replace (4.2) by

$$\delta_{n,p} = \begin{cases} \left((r_{\max})^{-nd} \sum_{x \sim_n y} |f(x) - f(y)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup\{|f(x) - f(y)| : x \sim_n y\} & \text{if } p = \infty, \end{cases} \tag{7.46}$$

we replace (4.3) by

$$\varepsilon_{n,p} = \begin{cases} \left((r_{\max})^{-nd} \sum_{x \in V_n \setminus V_0} |\Delta_n f(x)|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup\{|\Delta_n f(x)| : x \in V_n \setminus V_0\} & \text{if } p = \infty, \end{cases} \tag{7.47}$$

and we replace (4.4) by

$$\eta_{n,p} = \begin{cases} \left((r_{\max})^{-nd} \sum_{x \sim_n y} |\Delta_n f(x) - \Delta_n f(y)|^p \right)^{1/p} & \text{if } p < \infty, \\ \sup\{|\Delta_n f(x) - \Delta_n f(y)| : x \sim_n y\} & \text{if } p = \infty. \end{cases} \tag{7.48}$$

In (a) and (c) we do not know the appropriate upper bounds for α. Lemma 4.2 is again valid, but the other results are more technical and it is not clear in what way they extend.

In the special case when K is the unit interval we should ask whether or not the Besov spaces defined above coincide with the usual Besov spaces. Although we cannot answer this question in general, we can show at least one case where it is true: the space $(\Lambda_1^{2,2})^{(2)}(K)$ coincides with the usual Sobolev space L_1^2, which is identical to the usual $\Lambda_1^{2,2}$ Besov space. The reason for this is that our $(\Lambda_1^{2,2})^{(2)}(K)$ condition on f says exactly that the sum of the squares of the Haar coefficients of f' are finite, which is equivalent to $f' \in L^2$.

The results of Section 5 extend to the general setting with only minor modifications. The results of Section 6, on the other hand, rely on very specific symmetries of SG and so do not extend. Of course, Theorem 6.1 about restrictions is quite generic. It seems plausible that extension operators still exist, but other ideas will be required to describe them.

Acknowledgments

The author would like to thank Waldemar Hebisch, Takashi Kumagai, Laurent Saloff–Coste and Adam Sikora for discussions and communications about their work, and also Ricardo Saenz, who pointed out an error in an earlier version of this paper.

References

[Ba] M. Barlow, Diffusion on Fractals, in: Lecture Notes in Mathematics, Vol. 1690, Springer, Berlin, 1998.

[BP] M. Barlow, E. Perkins, Brownian motion on the Sierpinski gasket, Probab. Theory Related Fields 79 (1988) 543–623.

[BST] O. Ben-Bassat, R. Strichartz, A. Teplyaev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals, J. Funct. Anal. 166 (1999) 197–217.

[C] Z. Ciesielski, Approximation by splines and its application to Lipschitz classes and to stochastic processes, in: Teoria priblizenia funkcij, Proceedings of Conference in Kaluga 1975, Nauka, Moscow, 1977, pp. 397–404.

[DSV] K. Dalrymple, R. Strichartz, J. Vinson, Fractal differential equations on the Sierpinski gasket, J. Fourier Anal. Appl. 5 (1999) 203–284.

[DOS] X.T. Duong, E.M. Ouhabaz, A. Sikora, Plancherel type estimates and sharp spectral multipliers, preprint.

[HK] B. Hambly, T. Kumagai, Transition density estimates for diffusion processes on post critically finite self-similar fractals, Proc. London Math. Soc. 78 (3) (1999) 431–458.

[He] W. Hebisch, Functional calculus for slowly decaying kernels, preprint.

[HPS] P.E. Herman, PR. Strichartz, p-energy and p-harmonic functions on R. Peirone, Sierpinski gasket type fractals, preprint.

[J] A. Jonsson, Brownian motion on fractals and function spaces, Math. Z. 222 (1996) 495–504.

[JW] A. Jonsson, H. Wallin, Function spaces on subsets of $\mathbb{R}^n$, Math. Reports 2 (1984) 1–221.

[Ka] A. Kamont, A discrete characterization of Besov spaces, Approx. Theory Appl. 13 (1997) 63–77.

[Ki] J. Kigami, Analysis on Fractals, Cambridge University Press, Cambridge, 2001.

[Kum] T. Kumagai, Brownian motion penetrating fractals: an application of the trace theorem of Besov spaces, J. Funct. Anal. 170 (2000) 69–92.

[Mo] U. Mosco, Dirichlet Forms and Self–similarity, New Directions in Dirichlet forms, AMS/IP Studies in Advanced Mathematics, Vol. 8, Amer. Math. Soc. Providence, RI, 1998, pp. 117–155.

[SE] R.T. Seeley, Extension of C^∞ functions defined in a half space, Proc. Amer. Math. Soc. 15 (1964) 625–626.

[St] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.

[S1] R.S. Strichartz, Multipliers on fractional Sobolev spaces, J. Math. Mech. 16 (1967) 1031–1060.

[S2] R.S. Strichartz, A note on Trudinger's extension of Sobolev's inequalities, Indiana Univ. Math. J. 21 (1972) 841–842.

[S3] R. Strichartz, Analysis on fractals, Notices Amer. Math. Soc. 46 (1999) 1199–1208.

[SU] R. Strichartz, M. Usher, Splines on fractals, Math. Proc. Cambridge Philos. Soc. 129 (2000) 331.

[Tr] N.S. Trudinger, On imbedding into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–484.

[VSC] N.Th. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge University Press, Cambridge, 1992.